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Intermediate Algebra Overview

This Intermediate Algebra text provides the basic knowledge and skills necessary for success in college level mathematics courses. Our emphasis is on the study of structure and form, the relationships among various parts of algebra, the development of analytical reasoning, and the use of algebra as the language of mathematics. We place emphasis on understanding rather than on rote manipulation of symbols. We are striving for a mastery of a few basic principles underlying much of algebra; we believe that such mastery is the best preparation for grasping mathematics.

These materials do not approach topics in isolation, but integrate them, progressing from more basic ideas to more advanced ideas. Students learn best when they are actively involved with the mathematics they are studying, whether it be through explorations with technology, discovery activities, writing assignments, or extensive exercise sets. Therefore, students who practice communicating about mathematics in a variety of ways develop a more complete grasp of the concepts involved.
Preface to the Student

In order to succeed in this course, you need to know our ultimate goals so that you can focus on them. Our primary goal is for you to acquire an understanding of some fundamental principles of algebra. Memorizing “the rules” is not what we’re after. When you have a good grasp of the principles, the rules become natural and do not require rote memorization.

Secondly, we will be studying algebra itself; many of the concepts we discuss will be familiar, but we will discuss them in much greater depth. It is extremely important that you not fall into the trap of believing that you already know all about algebra. Because the topics are familiar, many students fall back on old (and bad) habits, and never learn algebra at the level they are capable of. Do not make this mistake! Please believe that you will not do well this semester unless you can clear your mind and start at the beginning with us.

We have divided the book into six units. The first unit discusses some common conventions, symbols, and terms that you should already be familiar with. The second unit is a detailed exploration of fractions; in particular, we examine why we treat them the way we do. The third unit is about what exponents are and how they are used. The fourth unit is about the distributive law. This law is one of the fundamental properties of our number system; understanding it will help you immensely in your algebra studies. The fifth unit is about lines, and the sixth unit is an introduction to functions. We approach all of the units in a fairly informal way, but we do make frequent use of symbols.

The class will not be easy. You will need to spend about three (3) hours outside of class for every hour in class; that means about 8 hours per week at a minimum. You may need more time; take that time! This is why college is a full-time job: a student taking 12 hours should be studying 24 to 36 hours outside of class, for a total of 36 to 48 hours.

If you finish your homework early, go over your notes, read the book, make up more problems for yourself. When you read the book, read it actively, not passively. Have a blank piece of paper, and work out all of the details for yourself even if they are in the book already. Cover them up and work through them again. We know that this is a lot of work, but if you are engaged at this level, you will do well. Just remember that a math text is not a novel; to read it properly, you must be actively involved. Mathematics is dense with symbols; each one carries a lot of meaning. A great deal of care is required to extract that meaning, but you can do it if you take the time. Commit to it now!

Jill Dumesnil and Colin Starr

May 20, 2013
Chapter 1

Preliminaries

1.1 Numbers and Equality

In this course, we will encounter several different number systems. We will briefly remind you of some of the more important ones here for future reference.

We are all familiar with the natural numbers: they are the usual counting numbers 1, 2, 3, 4, 5, 6, . . . . The whole numbers are the natural numbers together with 0: 0, 1, 2, 3, 4, 5, 6, . . . .

The integers are all of the whole numbers and their negatives:

\[ \ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots \]

The set of rational numbers is the set of ratios of integers (with nonzero denominators); that is, the set of fractions. Numbers such as

\[ \frac{2}{3}, \frac{17}{-5}, \frac{39}{13}, -6, \frac{-12}{-34} \]

are all rational numbers. There are several ways to represent any given rational number.

Notice that the rational numbers include the integers. For example, \(-6\) is a rational number since \(-6 = \frac{-6}{1}\). Also note that \(\frac{2}{3}\) is not a rational number.

*Example 1.1.1.* The fraction \(\frac{1}{4}\) is the same as 0.25, and can also be written as \(\frac{2}{8}, \frac{-1}{-4}\), and \(\frac{4}{16}\), among other ways.

The set of real numbers includes the set of all rational numbers and much more. Real numbers can be thought of as numbers that correspond to distances or lengths and the negatives of those numbers. For example, one can travel \(\frac{3}{10}\) of a mile, and one can travel \(\sqrt{2}\) miles. You may see a proof in a later course that \(\sqrt{2}\) is not a rational number: it cannot be written as a ratio of integers. However, it does correspond to a distance: it is the length of the diagonal of a square that is 1 unit long on each side.

Here are three important properties of equality that we will make frequent use of.
1. Equality is **reflexive**: every real number is equal to itself. That is, if \( a \) is a real number, then \( a = a \).

2. Equality is **symmetric**: if \( a = b \), then \( b = a \).

3. Equality is **transitive**: if \( a = b \) and \( b = c \), then \( a = c \).

*Example 1.1.2.* If we know that \( 0 = x - 4 \), we may rewrite this as \( x - 4 = 0 \) because equality is symmetric.

*Example 1.1.3.* If we know that \( x = 4 \) and \( 4 = y - 6 \), we may say that \( x = y - 6 \) because equality is transitive.

### Section 1.1 Exercises

Selected solutions to homework problems appear in the back of the book.

**True or False.**

1. 0 is a natural number.
2. \(-4\) is a whole number.
3. \(-12\) is an integer.
4. \(\frac{3}{4}\) is an integer.
5. \(\frac{7}{11}\) is a real number.
6. 0 is a whole number.
7. 7,412,294 is a natural number.
8. \(-47.3\) is an integer.

**Identify the property of equality that makes the statement true.**

9. Since \(\frac{8}{2} = 4\), \(4 = \frac{8}{2}\).
10. If \(x + 2 = y\) and \(y = 9\), then \(x + 2 = 9\).
11. \(3x = 3x\).
12. If \(11 = 7 + 8t\), then \(7 + 8t = 11\).
13. No matter real number what \(z\) is, \(\frac{z}{2} = \frac{z}{2}\).
14. If \(3(a + 1) = 3a + 3\), then \(3a + 3 = 3(a + 1)\).
15. If \(\frac{10}{15} = \frac{2}{3}\) and \(\frac{2}{3} = \frac{8}{12}\), then \(\frac{10}{15} = \frac{8}{12}\).
16. If \(x + 2 = 2 + x\), then \(2 + x = x + 2\).
1.2 Variables

In mathematics, we have a frequent need to refer to an unknown number. Consider the following situation.

**Example 1.2.1.** Timothy has three times as many eggs as Sarah. If Timothy has 8 more eggs than Sarah, how many eggs does each have?

We can answer this question as follows: 3 times the number of eggs Sarah has is the number of eggs Timothy has, and so is 8 more than the number of eggs Sarah has. Thus

\[ 3 \times (\text{the number of eggs Sarah has}) = (\text{the number of eggs Timothy has}) \]

and

\[ (\text{the number of eggs Timothy has}) = (\text{the number of eggs Sarah has}) + 8. \]

Using the transitive property of equality (above), we see that

\[ 3 \times (\text{the number of eggs Sarah has}) = (\text{the number of eggs Sarah has}) + 8. \]

If we subtract the number of eggs Sarah has from each side of the equation, we get

\[ 2 \times (\text{the number of eggs Sarah has}) = 8, \]

so Sarah must have 4 eggs. Therefore, Timothy has 12 eggs.

\[ \square \]

You may have noticed that writing “the number of eggs Sarah has” and “the number of eggs Timothy has” was rather awkward and unwieldy. To streamline the process, we choose a variable to represent each quantity. Let us choose \( s \) to represent the number of eggs Sarah has and \( t \) to represent the number of eggs Timothy has. The choices of names for our variables are completely arbitrary, so we picked names that would help us remember what each variable stands for: \( s \) for Sarah and \( t \) for Timothy. Now our equations above look like

\[ 3s = t \quad \text{and} \quad t = s + 8, \quad \text{so that} \quad 3s = s + 8. \]

Now we can subtract \( s \) from both sides to get \( 2s = 8 \) and \( s = 4 \). That is, Sarah has 4 eggs and Timothy must have \( 3 \cdot 4 = 12 \) eggs.

Using variables to represent the quantities made the equations much more compact. However, you must be careful to remember what each variable represents, too. If you introduce a new variable into a problem, you must specify carefully what quantity it is representing.

There is another inherent difficulty in introducing variables: it represents a step toward a more abstract way of thinking, which can be difficult at first. However, the power you gain from it is well worth the effort; we will see throughout this course that variables are extremely powerful and useful.

**Example 1.2.2.** The Thompsons needed to have sand delivered for a construction project at their home. The cost of the sand was $8 per cubic yard plus a $10 delivery charge. Express the total cost in terms of the number of cubic yards of sand they ordered.

**Solution:** We could express this as

\[ 8 \times (\text{the number of cubic yards of sand the Thompsons had delivered}) + 10, \]
but that is very awkward. Instead, let’s let $s$ represent the number of cubic yards of sand they had delivered. Then the cost of getting the sand was

$$8s + 10.$$

Section 1.2 Exercises

Choose variables to express each quantity.

1. James has to pay the municipal court a $10 fee for every petition he files. Express the amount of money James pays the court in terms of the number of petitions he files.

2. Carlos pays his bank a monthly fee of $4.95 plus a $0.50 surcharge every time he uses an ATM. Express Carlos’ bank fees in terms of the number of times he uses an ATM.

3. On a business trip, Kim pays a hotel tax of $9.90 every night in addition to the $85 she pays for her room every night. Express Kim’s hotel bill in terms of the number of nights she stays at the hotel.

4. For after-hour emergencies, a plumber charges a fee of $40 plus an hourly rate of $60. Express the plumber’s charges for an after-hours emergency call in terms of how many hours she works.

5. A gallon of paint covers about 400 square feet of wall space. Express the number of gallons necessary for a paint job in terms of the area that needs to be painted. [Be careful!]

6. It takes Susan about one hour to mow an acre of lawn. She also needs an hour to clean her equipment after she has finished. Express the amount of time it takes Susan to mow in terms of the number of acres she must mow.

Find a solution to each problem.

7. Carlos pays his bank a monthly fee of $4.95 plus a $0.50 surcharge every time he uses an ATM. If his fees one month totalled $8.45, how many times did he use an ATM?

8. It takes Susan about one hour to mow an acre of lawn. She also needs an hour to clean her equipment after she has finished. If a lawn takes her 12 hours to mow, including clean-up, how many acres is the lawn?

9. Pat works for 4 hours more than Juanita. This is also twice as long as Juanita worked. How long did each work?

10. On a business trip, Kim pays a hotel tax of $9.90 every night in addition to the $85 she pays for her room every night. If her total bill was $569.40, how many nights did she stay?
11. It takes Thadd 15 minutes to set up his equipment every day. Once it is set up, Thadd can make a designer pen in about 45 minutes. How many pens can Thadd make if he works for 7 hours?

1.3 Grouping Symbols

Before we begin, we need to recall some terminology from earlier mathematics courses. When numbers are added together, each of the numbers is called a term and the total is their sum. When numbers are multiplied together, each number is called a factor and the result is their product.

Example 1.3.1. In the sum $4 + xy + 19$, $4$ is a term, $xy$ is a term, and $19$ is a term. Within the term $xy$, $x$ is a factor and $y$ is a factor since they are multiplied together.

Example 1.3.2. In the expression $9 + 45 + 12$, the sum is $66$, and each of $9$, $45$, and $12$ is a term.

Example 1.3.3. In the expression $5 - 9$, the terms are $5$ and $-9$ since $5 - 9 = 5 + (-9)$. Thus, subtraction is a form of addition. We will make frequent use of this idea without necessarily referring to it explicitly, so make sure you understand it.

Example 1.3.4. In the product $x(x + 3)(y - 4)$, $x$ is a factor, $x + 3$ is a factor, and $y - 4$ is a factor. Within the factor $x + 3$, $x$ is a term and $3$ is a term. Within the factor $y - 4$, $y$ is a term and $-4$ is a term since $y - 4 = y + (-4)$.

The parentheses in the previous example are used as grouping symbols. We need grouping symbols to indicate the order of operations; that is, the order in which we want to perform the operations we are given. In general, operations within grouping symbols must be performed before what is grouped can be used in another computation.

Example 1.3.5. $(4 + 3) \cdot 9 = 7 \cdot 9$. The sum $4 + 3$ needs to be computed before we can multiply by $9$ because the sum $4 + 3$ is grouped within the parentheses. The parentheses indicate that what is inside them should be treated as a single number.

Parentheses are not the only grouping symbols; we will encounter many more in our explorations. Some common grouping symbols are $[\ ]$ and $\{\}$; they are used in the same way as parentheses. Usually, they are used to make a complicated expression easier to read.
Example 1.3.6. Compute $12 - [4 - (5 - 8)]$. The square brackets [ and ] indicate that we must compute $4 - (5 - 8)$ before we can subtract that from 12. The parentheses tell us that in order to compute $4 - (5 - 8)$, we must first compute $5 - 8 = -3$. Thus, we have

$$12 - [4 - (5 - 8)] = 12 - [4 - (-3)] = 12 - 7 = 5$$

Notice that the innermost parentheses dictated the first computation. Also notice that $12 - [4 - (5 - 8)]$ is easier to read than $12 - (4 - (5 - 8))$. The different grouping symbols help to guide our eyes.

\[  \]

Section 1.3 Exercises

Express each verbal description mathematically.

1. The product of 8 and 5.
2. The sum of 12 and 19.
3. The product of 12 and the sum of 3 and 5.
4. The sum of 4 and the product of 6 and 5.
5. Four times five less than eight.
6. Five less than four times eight.

Express each mathematical expression verbally using the terms “sum” and “product” as appropriate.

7. $4 \cdot 5$
8. $7 + 12 + 5$
9. $3 \cdot 9 + 15$
10. $6(5 - 2)$
11. $5 + 9(8 + 2)$
12. $12(6 + 1) + 4$

Compute as indicated.

13. $5 \cdot (4 + 2)$
14. $5 + [(5 - 7) \cdot (-3)]$
15. $6 - [5 \cdot (7 + 1)]$
16. $(4.6 + 12.2) + 6.8$
17. $4.6 + (12.2 + 6.8)$
18. $36 \div (9 + 3)$
19. $41 - (18 + 6)$
20. $7 + \{[4 - 6(8 + 1)] \cdot 2\}$
Rewrite each difference as a sum.

21. $7 - 12$
22. $14 - 8$
23. $x - 5$
24. $4 - 11$
25. $117 - x$
26. $12 - 4x$

### 1.4 Order of Operations

In the absence of grouping symbols, the expression

$$4 + 7 \cdot 8$$

could be ambiguous. Do we add the 4 and the 7 first, and then multiply by 8, or do we multiply 7 and 8 first and then add the result to 4? The first way gives us 88 and the second way gives us 60, so it will make a difference.

To avoid this ambiguity, we need to all agree on which operations come first. Mathematicians have adopted a convention: multiplication and division receive a higher priority than addition and subtraction; that is, they are performed first, as though the product had been written inside of grouping symbols.

*Example* 1.4.1. The second interpretation above is the correct one:

$$4 + 7 \cdot 8 = 4 + (7 \cdot 8)$$

$$= 4 + 56$$

$$= 60.$$  

\[\square\]

When there are several multiplications and/or divisions, they are performed from left to right; likewise, when there are several additions and/or subtractions, they are also performed from left to right.

*Example* 1.4.2.

$$6 \cdot 12 \div 9 \cdot 4 = [(6 \cdot 12) \div 9] \cdot 4$$

$$= [72 \div 9] \cdot 4$$

$$= 8 \cdot 4 = 32.$$  

\[\square\]

*Example* 1.4.3.

$$4 + 53 - 12 + 17 = [(4 + 53) - 12] + 17$$

$$= [57 - 12] + 17$$

$$= 45 + 17$$

$$= 62.$$  

\[\square\]
Example 1.4.4.

\[
7 + 5 \cdot 8 \div 2 - 14 = \{7 + [(5 \cdot 8) \div 2]\} - 14
\]
\[
= \{7 + [40 \div 2]\} - 14
\]
\[
= \{7 + 20\} - 14
\]
\[
= 27 - 14
\]
\[
= 13.
\]

Notice that even though the multiplication and division were in the middle of the expression, they were performed first.

It is important to realize that the order of operations described above is a choice: we all needed to agree on what symbols mean so that we could communicate effectively, and the decisions we have detailed above are what we have all agreed on. This means that in order for you to understand what you read mathematically, you must understand the order of operations. It is analogous to agreeing that in English, we read from left to right and not tfel ot thgir!

Finally, recall that we have several ways of writing multiplication: if \(a\) and \(b\) are numbers, then

\[
a \cdot b = ab = (a)(b) = a(b) = (a)b.
\]

Section 1.4 Exercises

Compute as indicated.

1. \(41 - 18 + 6\)
2. \(2.5 + 5 \cdot 8 - 3\)
3. \(17 - 8 - 5\)
4. \(5 \cdot 4 + 5 \cdot 2\)
5. \(28 \div 4 + 3\)
6. \(28 \div (4 + 3)\)
7. \((28 \div 4) + 3\)
8. \(6 + (5 - 8) \cdot [6 + 2(5 - 7)]\)
9. \(3 \cdot 3 \cdot 3 \cdot 3\)
10. \(4 \div (8 + 12) \cdot 80\)
11. \(8 - \{5 - [3 - (6 - 4)]\}\)
12. \((6 - 3) \div 2 + 1\)

Insert grouping symbols to explicitly indicate the order of operations.
13. $4 \cdot 3 + 1$
14. $4 + 5 \cdot 3 + 2$
15. $5 - 3 - 4$
16. $6 \cdot 7 \div 2$
17. $4 + 12 \div (2 \cdot 3) - 3 - 4$.
18. $8 + 2 + 5 - 3$
19. $4 \cdot 6 \div 2 \cdot 11$
20. $2 - 15 \div 3 + 2$

1.5 Properties of Operations

Based on our observations of the world around us, we have come to believe that certain things are true. For example, we expect that if we compute the sum $5 + 17$, we will arrive at the same result as if we compute the sum $17 + 5$; after all, if we combine a pile of 17 stones with another pile of 5 stones, we expect that it does not matter which pile we put in first. Either way, we expect the final pile to have 24 stones in it. (Just checking! 22 stones, of course.) These properties are given special names so that we can talk about them without saying, “that property that says that $5 + 17$ and $17 + 5$ are equal.” We list these names below.

**Theorem 1.** The following properties of operations hold.

1. **Addition of real numbers is commutative:** $a + b = b + a$ for all real number $a$ and $b$.
2. **Addition of real numbers is associative:** $(a + b) + c = a + (b + c)$ for all real numbers $a, b,$ and $c$.
3. **Multiplication of real numbers is commutative:** $a \cdot b = b \cdot a$ for all real number $a$ and $b$.
4. **Multiplication of real numbers is associative:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all real numbers $a, b,$ and $c$.
5. **Multiplication distributes over addition:** $a(b + c) = ab + ac$ for all real numbers $a, b,$ and $c$.
6. There is an **identity** for addition: if $a$ is any real number, then $a + 0 = 0 + a = a$.
7. There is an **identity** for multiplication: if $a$ is any real number, then $a \cdot 1 = 1 \cdot a = a$.
8. **Every real number has an additive inverse:** if $a$ is a real number, then
   
   $$a + (-a) = (-a) + a = 0.$$ 

9. **Every real number except 0 has a multiplicative inverse:** if $a$ is a real number, then
   
   $$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.$$ 

We will explore the distributive property in great detail in Chapter 4.
Example 1.5.1. Computation yields that $(23.5) \cdot 9 = 211.5$ and $9 \cdot (23.5) = 211.5$. This illustrates the commutativity of multiplication.

Example 1.5.2. Note that $[5 + (-3)] + 6 = 2 + 6 = 8$, and $5 + (-3 + 6) = 5 + 3 = 8$. This illustrates the associativity of addition.

Example 1.5.3. Since 0 is the additive identity, we have that $4127 + 0 = 4127$ and $0 + 4127 = 4127$.

Example 1.5.4. Since $8 \cdot (0.125) = (0.125) \cdot 8 = 1$, we see that 8 is the multiplicative inverse of 0.125 and 0.125 is the multiplicative inverse of 8.

Section 1.5 Exercises

Identify which property above is used to justify each statement.

1. $-3.751 + 0 = -3.751$
2. $(6 \cdot 5) \cdot 2 = 6 \cdot (5 \cdot 2)$
3. $3x + 6x = (3 + 6)x$
4. $42 \cdot \frac{1}{42} = 1$
5. $7 + (-7) = 0$
6. $0 + y = y$
7. $1 \cdot x = x$
8. $(x + y) \cdot (a + b) = (a + b) \cdot (x + y)$
9. $(x + y)a + (x + y)b = (x + y)(a + b)$
10. $a + b = b + a$
11. $a + (-a) = 0$
12. $(x + 2)\frac{1}{(x+2)} = 1$

13. A customer in a back aisle of your store has seven items worth $1.52 each. She tells you that her partner will soon be arriving with three more of the same item. What field property allows you to quickly compute the total value of the items, $7 \cdot 1.52 + 3 \cdot 1.52$?

14. Identify which field property is used in each step below.

\[
(x + 2)(x + 3) = [(x + 2) \cdot x] + [(x + 2) \cdot 3]
\]
\[
= [x \cdot (x + 2)] + [3 \cdot (x + 2)]
\]
\[
= [x^2 + 2x] + [3x + 6]
\]
\[
= [(x^2 + 2x) + 3x] + 6
\]
\[
= [x^2 + (2 + 3)x] + 6
\]
\[
= x^2 + 5x + 6
\]

\[
2 + 3 = 5
\]
15. In algebra, we often have a need to “combine like terms.” For example, \(7x + 5x = 12x\). What property of operations allows us to combine like terms? Show that \(7x + 5x = 12x\) using the properties of operations.

### 1.6 Sets

We often have collections of objects that we need to describe in some compact way. In this section, we give a brief introduction to the idea of a set.

A set is a collection of objects described unambiguously. Thus, “the collection of people over 6 feet tall” is a set, but “the collection of tall people” is not a set. Most of the time, we will be interested in sets of numbers, but we will not restrict ourselves to them.

We have two main ways of writing sets. The first is roster notation.

**Example 1.6.1.** The set of natural numbers less than 7 is written in roster notation as \(\{1, 2, 3, 4, 5, 6\}\). The “curly braces” \(\{\) and \(\}\) indicate where the set begins and ends.

\[\square\]

**Example 1.6.2.** The set of numbers satisfying the equation \(x^2 = 4\) is \(\{2, -2\}\).

\[\square\]

**Example 1.6.3.** The set of natural numbers is \(\{1, 2, 3, 4, 5, 6, \ldots\}\). The dots \(\ldots\) indicate that the established pattern is to be continued.

\[\square\]

Our second notation is called set-builder notation; we use it when sets are too big or too complicated for us to actually list all of the elements or even establish a pattern.

**Example 1.6.4.** Consider the set consisting of all real numbers except zero. We can’t list them all, and we can’t even describe a nice pattern. However, we have a convenient shorthand way to describe this set, too. We write

\[\{x | x \neq 0\},\]

where again the curly braces \(\{\) and \(\}\) indicate the beginning and ending of the set. Here are what the symbols all mean.

\[
\{ x \mid x \neq 0 \}
\]

The set of all \(x\) such that \(x\) is not equal to 0.

The vertical bar comes right before the condition that determines whether or not \(x\) gets to be in the set. That is, the condition right after the vertical bar is the “entrance exam” \(x\) must pass to get into the set.

\[\square\]

**Example 1.6.5.** \(\{x | x \geq 0\}\) is the set of all real numbers that are zero or greater.
Example 1.6.6. \( \{x | x \neq 1, 2, 3, 4\} \) is the set of all real numbers except 1, 2, 3, and 4.
Chapter 2

Fractions

Recall from Chapter 1 that the **rational numbers** are those numbers that can be written as a ratio of integers (with nonzero denominators); they are the **fractions**. We will develop our usual rules for adding, subtracting, multiplying, and dividing fractions by looking at the models that illustrate these concepts.

### 2.1 The concept of a fraction

Most of us encounter the natural numbers very early on; we see that a sibling has two pieces of candy, for example, while we only have one. However, it is not long before we discover that the natural numbers are insufficient to describe the world around us. We need negative numbers to discuss debt, loss of yardage in football, temperature, and so on. The next step is to use numbers to describe parts of things, and that is what this chapter is about.

We will begin by considering a whole object; for example, a pie. Suppose we cut that pie into eight equal pieces, as illustrated.

![Figure 2.1: A pie cut into 8 equal pieces](image)

We say that each piece is one **eighth** of the pie. The eight equal pieces together comprise the whole pie; that is, if we take eight eighths, we have one whole. We represent these quantities with **fractions**, which we represent with symbols of the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \neq 0 \).

The fraction \( \frac{1}{8} \) carries two pieces of information. The 8 in the **denominator** indicates the number of equal pieces the whole has been cut into. The **numerator** indicates the number of those pieces we have. Thus, the fraction \( \frac{3}{8} \) indicates that the pie has been cut into 8 equal pieces, and we have 3 of those. In Figure 2.2, the shaded portion is the portion we have.
This is the basic premise of fractions. Fractions are another way of considering division since we begin by dividing a unit into equal pieces. In fact, you may have noticed that the division symbol, \( \div \), looks like a fraction bar with little dots for placeholders in the numerator and denominator.

There are many models for fractions; in the examples below, we consider some of these.

**Example 2.1.1.** The rectangle below has been divided into 10 equal pieces, 4 of which are shaded. The shaded portion therefore represents \( \frac{4}{10} \).

![Figure 2.3: \( \frac{4}{10} \)](image)

This illustrates a rectangular model of fractions.

**Example 2.1.2.** The following rectangular model represents the fraction \( \frac{2}{5} \).

![Figure 2.4: \( \frac{2}{5} \)](image)

**Example 2.1.3.** Some fractions consist of more than one whole unit. The figure below illustrates the fraction \( \frac{17}{12} \).
This can also be written as \(1 + \frac{5}{12}\), since it is a whole unit plus \(\frac{5}{12}\) more. Notice that the unit is one strip, and it has been divided into 12 equal pieces. Since we have 17 of those pieces shaded, the fraction represented is \(\frac{17}{12}\).

Sometimes a fraction such as \(\frac{17}{12}\) will appear as a **mixed number**:

\[
\frac{17}{12} = 1 \frac{5}{12}.
\]

When a fraction is written this way, it means

\[
1 + \frac{5}{12}.
\]

Because the mixed number notation looks like multiplication, our use of mixed numbers in this text will be very limited.

This model illustrates a **strip model** of fractions.

---

We will now adopt a more formal definition of a fraction.

**Definition 2.1.1.** A **rational number** is a number that can be written in the form \(\frac{a}{b}\), where \(a\) and \(b\) are integers and \(b \neq 0\). The number \(a\) is called the **numerator** and the number \(b\) is called the **denominator**.

We cannot allow \(b\) to equal 0 because it makes no sense to divide an object into 0 pieces!

The **denominator** tells you the size of the equal pieces, just as the **denomination** of paper money tells you its value (e.g., $10, $5, etc.). The **numerator** tells you the **number** of those pieces.

**Example 2.1.4.** We have seen some example of rational numbers in Chapter 1. Here are some more examples of rational numbers:

\[
\frac{3}{7}, \frac{-29}{13}, \frac{40}{20}, 0, \frac{-12}{-48}.
\]

---

**Example 2.1.5.** The famous number \(\pi \approx 3.14159\) is **not** a rational number; it cannot be written as \(\frac{a}{b}\) with \(a\) and \(b\) integers. The digits to the right of the decimal neither terminate nor repeat; proving this requires techniques that are beyond the scope of this course.
Example 2.1.6. The symbol $\frac{3}{0}$ does not represent a rational number since the denominator is 0.

Example 2.1.7. For the symbol $\frac{1}{x}$ to represent a rational number, we must have $x \neq 0$.

Example 2.1.8. For the symbol $\frac{12}{x+3}$ to represent a rational number, we must have $x \neq -3$.

Example 2.1.9. If $a$ is any integer, then the fraction $\frac{a}{1}$ is just equal to $a$. Why is this? The 1 in the denominator indicates that we are taking a unit and dividing into 1 piece; that is, we aren’t dividing it at all. Since we then take $a$ of those unit-sized pieces, we have a total of $a$ units!

Thus,

$$\frac{3}{1} = 3, \quad \frac{-12}{1} = -12, \quad \text{and} \quad \frac{0}{1} = 0.$$

Example 2.1.10. Percentages are special examples of fractions. A percentage refers to a fraction with a denominator of 100. Thus,

$$14\% = \frac{14}{100}, \quad \text{and} \quad 39.6\% = \frac{39.6}{100}.$$

This last doesn’t look like a fraction in the sense we have discussed, but we can write it in the appropriate form by multiplying numerator and denominator by 10:

$$\frac{39.6 \cdot 10}{100 \cdot 10} = 0.396.$$

Remember that fractions refer to division. This means that $\frac{4}{5}$ is the same as $4 \div 5$. We can use this division idea to find decimal equivalents of fractions.

Example 2.1.11. Since $4 \div 5 = 0.8$, we see that $\frac{4}{5} = 0.8$.

Example 2.1.12. Some well-known decimal equivalents are

$$\frac{1}{2} = 0.5, \quad \frac{1}{4} = 0.25, \quad \frac{3}{4} = 0.75, \quad \frac{1}{3} = 0.\overline{3}, \quad \text{and} \quad \frac{3}{10} = 0.3,$$

where the $0.\overline{3}$ represents $0.3333333\ldots$. 
Section 2.1 Exercises

Draw a figure representing each fraction. Be sure to identify 1 unit.

1. \( \frac{3}{4} \)  
2. \( \frac{2}{5} \)  
3. \( \frac{4}{3} \)  
4. \( \frac{4}{6} \)  
5. \( \frac{8}{2} \)  
6. \( \frac{9}{10} \)

Find a fraction that is equal to each decimal.

7. 0.25  
8. 0.4  
9. 1.5  
10. 0.6

Identify the fraction represented by each model.

11.  
12.  
13. 1 unit  
14.  
15.  
16.  

Convert each fraction to a percentage.

17. \( \frac{45}{100} \)  
18. \( \frac{72}{100} \)  
19. \( \frac{16}{50} \)  
20. \( \frac{28}{25} \)  
21. \( \frac{2}{5} \)  
22. \( \frac{5}{6} \)

Convert each percentage to a fraction.
23. 14%  
24. 27%  
25. 138%  
26. 0.45%  
27. 1%  
28. 12.5%

Express each decimal as a fraction.

29. 0.25  
30. 0.4  
31. 0.6  
32. 1.5  
33. 0.7  
34. 1.3

\section{2.2 Fraction Equivalence}

Consider the fraction $\frac{6}{8}$ back in Figure 2.2. Below is another fraction, $\frac{3}{4}$, that appears to have the same area shaded as $\frac{6}{8}$. In the second figure, we have superimposed the $\frac{6}{8}$ picture on top of the $\frac{3}{4}$ picture.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fraction_equivalence.png}
\caption{\(\frac{3}{4} = \frac{6}{8}\)}
\end{figure}

The shaded regions show that $\frac{6}{8} = \frac{3}{4}$. Let’s think about what the symbols all mean. The symbol $\frac{6}{8}$ means that we have divided a unit into 8 equal pieces and taken 6. Notice that if we gather the 8 equal pieces into groups of 2 pieces each, we will have exactly 4 groups, all of equal size. Of those, we have a total of 3 groups of 2 pieces each that are shaded. That is exactly what $\frac{3}{4}$ means: we have divided the unit into 4 equal pieces and taken 3!

This concept is known as \textbf{fraction equivalence}. Intuitively, two fractions are equivalent if they represent the same quantity.

If two fractions have the same denominator, then the fractions are equivalent if and only if they have the same numerator.

\textit{Example 2.2.1.} The fractions $\frac{4}{15}$ and $\frac{7}{15}$ have the same denominator; therefore, they are not equivalent because their numerators are not the same.

What if two fractions have different denominators?

\textit{Example 2.2.2.} The fraction strip shown represents $\frac{2}{4}$ since the unit, one strip, has been divided into 4 equal pieces and 2 of those are shaded.
Without changing the portion shaded, we can subdivide those 4 pieces into 3 pieces each, so we end up with a total of $4 \cdot 3 = 12$ pieces. Originally 2 pieces were shaded, and we see that now $2 \cdot 3 = 6$ pieces are shaded, so we have $\frac{6}{12}$ shaded. Thus, $\frac{2}{4}$ and $\frac{6}{12}$ are equivalent fractions.

This illustrates a general principle.

**Theorem 2** (Fraction Simplification). *Let $\frac{a}{b}$ be a fraction and $n$ be a nonzero integer. Then*

$$\frac{a}{b} = \frac{a \cdot n}{b \cdot n}.$$  

This theorem works because of the idea of subdivision just discussed in Example 2.2.2. The fraction $\frac{a}{b}$ indicates that we have divided a unit into $b$ equal pieces and have $a$ of them (shaded). If we then divide each of those into $n$ pieces, we end up with a total of $b \cdot n$ pieces, of which $a \cdot n$ are shaded. This means that the original fraction $\frac{a}{b}$ is equivalent to $\frac{a \cdot n}{b \cdot n}$. We will see in the next section why this theorem is named the Fraction Simplification Theorem.

**Example 2.2.3.** The fractions $\frac{1}{3}$ and $\frac{5}{15}$ are equivalent since

$$\frac{1}{3} = \frac{1 \cdot 5}{3 \cdot 5} = \frac{5}{15},$$

as the figure illustrates.
Example 2.2.4. The fractions $\frac{4}{6}$ and $\frac{2}{3}$ are equivalent since both of them are equal to $\frac{2}{3}$. Also, we may write $\frac{4}{6}$ as $\frac{2 \times 2}{3 \times 2}$.

Example 2.2.5. The fractions $\frac{7}{3}$ and $\frac{42}{18}$ are equivalent since

$$\frac{7}{3} = \frac{7 \cdot 6}{3 \cdot 6} = \frac{42}{18}.$$

□

Example 2.2.6. This idea works even when the numbers are not known; for example,

$$\frac{x}{x+1}$$

is equivalent to

$$\frac{4x}{4(x+1)}.$$  

Note that we must have the implicit assumption that $x \neq -1$ for the fractions in this example to be defined.

□

Example 2.2.7. Show that $\frac{8}{12}$ and $\frac{10}{15}$ are equivalent.

Solution: We have seen that it is easy to compare fractions with the same denominator. Our strategy, therefore, will be to find a fraction equivalent to $\frac{8}{12}$ and another fraction equivalent to $\frac{10}{15}$, where both new fractions have the same denominator. One way to get a common denominator is to multiply each denominator by the other: since $12 \cdot 15 = 15 \cdot 12$, the new denominators will be equal.

$$\frac{8}{12} = \frac{8 \cdot 15}{12 \cdot 15} = \frac{120}{12 \cdot 15} \quad \text{and} \quad \frac{10}{15} = \frac{10 \cdot 12}{15 \cdot 12} = \frac{120}{12 \cdot 15}.$$

Now these two fractions have the same denominator ($15 \cdot 12$) and both have a numerator equal to 120. Therefore, they are equivalent.

We can also use a rectangular array model to see that these are equivalent. Below are models for $\frac{8}{12}$ and $\frac{10}{15}$.

![Figure 2.10: $\frac{8}{12}$ and $\frac{10}{15}$](image-url)
We will subdivide the model for $\frac{8}{12}$ into 15 equal pieces by using horizontal lines. We will also subdivide the model for $\frac{10}{15}$ into 12 equal pieces by using vertical lines. Remember, this doesn’t change at all the amount of the diagram that is shaded.

Both rectangles are the same size, and both are divided into $15 \cdot 12 = 12 \cdot 15 = 180$ equal pieces. In the diagram for $\frac{8}{12}$, we have $8 \cdot 15 = 120$ pieces shaded. In the diagram for $\frac{10}{15}$, we have $10 \cdot 12 = 120$ pieces shaded. Thus, these fractions are equivalent.

In this example, we chose a factor to multiply the numerator and denominator of each side by. We chose the factor for the left-hand side to be the right-hand side’s denominator, and vice-versa.

This illustrates a pattern that appears in the Fraction Simplification Theorem, Theorem 2.2.1.

Example 2.2.8. Suppose that

$$\frac{a}{b} = \frac{c}{d},$$

where $\frac{a}{b}$ and $\frac{c}{d}$ are two given fractions. Following the same idea we had above, we will find common denominators so that we can compare numerators.

$$\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$ 

Thus, because the two fractions are equal and now have the same denominator, we must have $ad = bc$.

This idea is often taken as the definition of fraction equivalence; here, we will present it as a theorem.

**Theorem 3** (Fraction Equivalence). *Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$.***

Example 2.2.9. The fractions $\frac{4}{6}$ and $\frac{2}{3}$ are equivalent since $4 \cdot 3 = 12 = 6 \cdot 2$. In terms of the Fraction Simplification Theorem, this looks like

$$\frac{4}{6} = \frac{4 \cdot 3}{6 \cdot 3} \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 6}{3 \cdot 6}.$$
Then we are just comparing the numerators $4 \cdot 3$ and $6 \cdot 2$, which is precisely what the theorem tells us to do!

Example 2.2.10. The fractions $\frac{3}{6}$ and $\frac{4}{9}$ are not equivalent since $3 \cdot 9 \neq 4 \cdot 6$. Also, $\frac{3}{6}$ is 0.5 and $\frac{4}{9} = 0.444\ldots$ is less than 0.5.

Example 2.2.11. The fractions $\frac{16}{30}$ and $\frac{25}{45}$ are not equivalent since $16 \cdot 45 = 720$ and $30 \cdot 25 = 750$.

Example 2.2.12. The fractions $\frac{4}{5}$ and $\frac{-4}{5}$ are equivalent since $4(-5) = (-4)(5)$.

Example 2.2.13. For what integer $x$ will $\frac{2}{3} = \frac{12}{18}$?

Solution 1. We can rewrite the fraction $\frac{2}{3}$ with denominator 18 by

$$\frac{x}{3} = \frac{x \cdot 6}{3 \cdot 6} = \frac{6x}{18}.$$ 

Thus for $\frac{6x}{18} = \frac{12}{18}$, we’d need that $6x = 12$ so that $x = 2$.

Solution 2. Using the Fraction Equivalence Theorem, we’d have $\frac{2}{3} = \frac{12}{18}$ if and only if $x \cdot 18 = 3 \cdot 12$. Thus, we’d need $18x = 36$ so that $x = 2$.

Section 2.2 Exercises

In each case, show that the fractions are equivalent using (a) the Fraction Simplification Theorem and (b) the Fraction Equivalence Theorem.

1. $\frac{2}{3}$ and $\frac{8}{12}$  
2. $\frac{6}{15}$ and $\frac{2}{5}$  
3. $\frac{9}{12}$ and $\frac{3}{4}$  
4. $\frac{5}{6}$ and $\frac{25}{30}$  
5. $\frac{-4}{8}$ and $\frac{-1}{2}$  
6. $\frac{45}{60}$ and $\frac{15}{20}$  
7. $\frac{28}{40}$ and $\frac{14}{20}$  
8. $\frac{33}{35}$ and $\frac{3}{5}$  
9. $\frac{-5}{8}$ and $\frac{5}{8}$  
10. $\frac{-3}{7}$ and $\frac{3}{-7}$  
11. $\frac{-21}{15}$ and $\frac{-7}{-5}$  
12. $\frac{4(x+y)}{11(x+y)}$ and $\frac{4}{11}$
Show that the given fractions are equivalent using the Fraction Equivalence Theorem.

13. \( \frac{12}{15} \) and \( \frac{8}{10} \)  
14. \( \frac{4}{5} \) and \( \frac{20}{30} \)  
15. \( \frac{12}{16} \) and \( \frac{9}{12} \)  
16. \( -\frac{110}{35} \) and \( -2 \)  
17. \( \frac{30}{35} \) and \( \frac{12}{14} \)  
18. \( \frac{121}{44} \) and \( \frac{33}{12} \)  
19. \( \frac{-4}{15} \) and \( \frac{-4}{15} \)  
20. \( \frac{-24}{31} \) and \( \frac{24}{31} \)  
21. \( \frac{x+1}{3(x+1)} \) and \( \frac{1}{3} \)  
22. \( \frac{1}{8} \) and \( \frac{5}{80} \)  
23. \( \frac{5x}{x} \) and \( 5 \)  
24. \( \frac{37}{37} \) and \( 37 \)

Each diagram illustrates a certain fraction. Subdivide each diagram to show that the fraction it represents is equivalent to the given fraction.

25. \( \frac{9}{12} \)  
26. \( \frac{12}{15} \)  
27. \( \frac{8}{20} \)  
28. \( \frac{10}{14} \)  
29. \( \frac{18}{48} \)  
30. \( \frac{32}{72} \)

31. Explain why \( \frac{a}{ab} = \frac{1}{b} \) for any integers \( a \) and \( b \), provided \( b \neq 0 \).
32. Explain why \( \frac{ab}{b} = a \) for any integers \( a \) and \( b \), provided \( b \neq 0 \).

### 2.3 Fraction Simplification

Recall from Chapter 1 that equality is symmetric; that is, if \( x = y \), then \( y = x \). This means that we can use the Fraction Simplification Theorem to simplify fractions: since

\[
\frac{a}{b} = \frac{an}{bn}, \quad \text{we also have} \quad \frac{an}{bn} = \frac{a}{b}.
\]

**Example 2.3.1.** Simplify the fraction \( \frac{4}{18} \).

**Solution:** We need to find factors that the numerator and denominator have in common. Since both are even, both have a factor of 2, and we can rewrite the fraction as

\[
\frac{4}{18} = \frac{2 \cdot 2}{9 \cdot 2} = \frac{2}{9},
\]

using the Fraction Simplification Theorem for the last step. Since 2 and 9 have no common integer factors, this is as far as we can simplify.
Notice that before we can simplify, we must factor both the numerator and denominator.

**Definition 2.3.1.** A fraction is said to be in **lowest terms** or **simplified** if the numerator and denominator have no common factors other than \( \pm 1 \).

We often want to express a given fraction in lowest terms.

**Example 2.3.2.** Simplify the fraction \( \frac{30}{96} \).

**Solution:** Since \( 30 = 5 \cdot 6 \) and \( 96 = 16 \cdot 6 \), we have

\[
\frac{30}{96} = \frac{5 \cdot 6}{16 \cdot 6} = \frac{5}{16}.
\]

Notice that we factored out the greatest common divisor (gcd) of 30 and 96.

**Example 2.3.3.** Simplify \( \frac{36}{48} \).

**Solution:**

\[
\frac{36}{48} = \frac{3 \cdot 12}{4 \cdot 12} = \frac{3}{4}.
\]

**Example 2.3.4.** Express \( \frac{-64}{24} \) in lowest terms.

**Solution:** Don’t let the minus sign throw you! We are still looking for common factors.

\[
\frac{-64}{24} = \frac{-8 \cdot 8}{3 \cdot 8} = \frac{-8}{3}.
\]

**Example 2.3.5.** Since multiplication is commutative (\( 3 \cdot 2 = 2 \cdot 3 \), for example), we do not need to have the fraction we want to simplify in exactly the form \( \frac{a \cdot n}{b \cdot n} \). For example,

\[
\frac{2 \cdot 3}{5 \cdot 2} = \frac{3}{5}
\]

since we could have rewritten the numerator as \( 3 \cdot 2 : \)

\[
\frac{2 \cdot 3}{5 \cdot 2} = \frac{3 \cdot 2}{5 \cdot 2}.
\]

Now the fraction is in the form of the Fraction Simplification Theorem, and we again get \( \frac{3}{5} \).

The previous example illustrates a useful time-saving device: if the numerator and denominator share a common factor at all (in any order), it may be removed from both.
Example 2.3.6. We have that
\[
\frac{2 \cdot 2 \cdot 5 \cdot 7}{5 \cdot 2 \cdot 3 \cdot 7} = \frac{2}{3},
\]
since the numerator and denominator share factors of 2, 5, and 7.

\[\square\]

Example 2.3.7. Simplify the fraction \( \frac{x \cdot (x - 2)}{(x - 2)(x + 3)} \).

**Solution.** First note that for the fraction to make sense, the denominator must not equal zero. Thus we have the implicit assumptions that \( x \neq 2 \) and \( x \neq -3 \).

Using the Fraction Simplification Theorem, we have
\[
\frac{x \cdot (x - 2)}{(x - 2)(x + 3)} = \frac{x \cdot (x - 2)}{(x + 3)(x - 2)} = \frac{x}{x + 3}, \quad x \neq 2.
\]

Note that in all of the above fractions we have the implicit assumption that \( x \neq -3 \), but for the final expression to equal the previous two expressions, we need the explicit assumption that \( x \neq 2 \) since it is no longer clear that this restriction applies.

\[\square\]

Example 2.3.8. Simplify the fraction \( \frac{256x}{120x^2} \).

**Solution.** In order to apply the Fraction Simplification Theorem, we must have the numerator and denominator factored:
\[
\frac{256x}{120x^2} = \frac{4 \cdot 2 \cdot 32 \cdot x}{4 \cdot 2 \cdot 15 \cdot x \cdot x} = \frac{32}{15 \cdot x} = \frac{32}{15x}
\]

Note that the assumption that \( x \neq 0 \) is implicit in all of the expressions above.

\[\square\]

The moral of this section is twofold: first, fraction equivalence arises from the very natural context of subdivision. Second, before you can simplify, you must factor!

**Section 2.3 Exercises**

Simplify each fraction using the Fraction Simplification Theorem. Be sure to include any implicit or explicit assumptions you may need.

1. \( \frac{4}{6} \)
2. \( \frac{8}{16} \)
3. \( \frac{6}{15} \)
4. \( \frac{21}{28} \)
5. \( \frac{24}{30} \)
6. \( \frac{16}{30} \)
7. \( \frac{45}{35} \)
8. \( \frac{66}{44} \)
9. \( \frac{36}{12} \)
10. \(\frac{96}{18}\)  
11. \(\frac{144}{60}\)  
12. \(\frac{39}{65}\)  
13. \(\frac{100}{84}\)  
14. \(\frac{-52}{39}\)  
15. \(\frac{14}{-91}\)  
16. \(\frac{-60}{-135}\)  
17. \(\frac{2xy}{8x}\)  
18. \(\frac{9(x + 1)(x - 2)}{15(x - 2)}\)  
19. \(\frac{51}{85}\)  
20. \(\frac{63z(x + 4)}{7(x + 4)}\)  
21. \(\frac{2 \cdot 2 \cdot 5 \cdot 11}{2 \cdot 7 \cdot 3 \cdot 11}\)  
22. \(\frac{4 \cdot 9 \cdot 25}{2 \cdot 3 \cdot 5}\)  
23. \(-\frac{21 \cdot 35}{(-35)(-21)}\)  
24. \(\frac{24x(y + 2)(z - 1)}{16x(z + 2)(y - 1)}\)  
25. \(\frac{40(4z - 7)}{(60)(8z - 14)}\)  
26. \(\frac{12x(x + 1)(x + 2)}{18(x + 1)x}\)  
27. \(\frac{8x - 6}{12x - 9}\)  
28. \(\frac{5t + 5}{25}\)  
29. \(\frac{4 + 6x}{4}\)  
30. \(\frac{12 - 6x}{2 - x}\)  
31. \(\frac{36x + 40}{x(27x + 30)}\)  
32. \(\frac{56(22x + 30)}{64}\)

33. At Rita’s high school, 600 out of 900 students are female. What (simplified) fraction of the students are female? What percentage of the students are female?

34. In a survey, 800 out of 1000 dentists preferred NoCav brand toothpaste to the alternative. What (simplified) fraction of dentists preferred NoCav? What percentage of dentists preferred NoCav?

### 2.4 Operations on Fractions: Addition and Subtraction

How should we add fractions? Perhaps the best way to answer this question is to consider what we mean by addition. Usually, we mean something like, if we start with 4 oranges and gain 5 oranges, how many do we have all together? Addition of fractions means something similar: if we eat \(\frac{3}{8}\) of a pie and then eat another \(\frac{2}{8}\) of a pie, what fraction of a pie do we have? Let’s consider this example.

*Example 2.4.1.* Figures for \(\frac{3}{8}\) and \(\frac{2}{8}\) are shown below. Remember what these fractions mean: the pie has been divided into 8 equal pieces; We begin with 3 of those and then gain 2 more, so we end up with 5; the figure supports this.
This indicates how we define addition of fractions that have a common denominator.

Definition 2.4.1. Let \( \frac{a}{c} \) and \( \frac{b}{c} \) be fractions. Then

\[
\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.
\]

Example 2.4.2. This time we will apply the definition above and look at a fraction strip model, as well. The exercise is to add \( \frac{5}{9} \) and \( \frac{2}{9} \). According to the definition, this should be

\[
\frac{5}{9} + \frac{2}{9} = \frac{5+2}{9} = \frac{7}{9}.
\]

To see that this is reasonable, consider the fraction strip models for \( \frac{5}{9} \) (top strip) and \( \frac{2}{9} \) (bottom strip). We have lined them up so that we can just count the shaded portions from left to right.

Since there are exactly 7 shaded portions, the total value is \( \frac{7}{9} \).

Example 2.4.3. The same idea holds when the sum is greater than a whole unit.

\[
\frac{7}{12} + \frac{9}{12} = \frac{7+9}{12} = \frac{16}{12}.
\]

In the figure, \( \frac{7}{12} \) is represented by the top strip, and \( \frac{9}{12} \) is represented by the bottom strip.
Notice that the shaded portion extends four squares beyond the full unit, so
\[
\frac{16}{12} = 1 + \frac{4}{12}.
\]

\[\square\]

**NOTE:** It is important to recognize that the fraction bar is also another grouping symbol. When you see
\[
\frac{a}{b},
\]
you should think of it as
\[
\frac{(a)}{(b)}.
\]
That is, the numerator is a group, the denominator is a group, and the fraction itself is a group. Thus,
\[
\frac{3 + 9}{16} \text{ is } \frac{(3 + 9)}{16},
\]
and not
\[
\frac{3}{16} + 9 \text{ or } 3 + \frac{9}{16}.
\]
The addition must be performed before the division because of the grouping.

**Example 2.4.4.** So far we have only considered positive fractions, but the definition of addition applies whether the fractions are positive or negative.
\[
\frac{-4}{7} + \frac{-6}{7} = \frac{(-4) + (-6)}{7} = \frac{-10}{7}.
\]
\[\square\]

**Example 2.4.5.**
\[
\frac{-5}{11} + \frac{9}{11} = \frac{(-5) + 9}{11} = \frac{4}{11}.
\]
\[\square\]

Since subtraction is really a form of addition, we have the following theorem about subtraction of fractions that have the same denominator.

**Theorem 4.** Let \( \frac{a}{c} \) and \( \frac{b}{c} \) be fractions. Then \( \frac{a}{c} - \frac{b}{c} = \frac{a - b}{c} \).

**Example 2.4.6.** Consider \( \frac{5}{6} - \frac{2}{6} \). First, we notice that we have divided the unit into 6 equal pieces. The subtraction indicates that we begin with 5 of these and take away 2, so we expect to be left with 3. The figure below illustrates this, and the computation using the theorem agrees as well. In the figure, the diagonal slashes represent the 2 shaded regions we are removing.
Figure 2.15: \( \frac{5}{6} - \frac{2}{6} = \frac{3}{6} \)

\[
\frac{5}{6} - \frac{2}{6} = \frac{5-2}{6} = \frac{3}{6} = \frac{1}{2}.
\]

Example 2.4.7.

\[
\frac{5}{8} - \frac{-3}{8} = \frac{5-(-3)}{8} = \frac{8}{8} = 1.
\]

What do we do when the denominators are different?

Example 2.4.8. If we want to add \( \frac{3}{7} \) and \( \frac{2}{5} \), we could use a strip model as above. To do this, we need a strip (one unit) divided up into 7 equal pieces and another strip of the same size as the first divided up into 5 equal pieces.

Figure 2.16: \( \frac{3}{7} \) and \( \frac{2}{5} \)

For a final answer, we need to know the total length of the shaded portions, but since the shaded pieces are of different sizes, it is very difficult to tell what that length is! We really need to arrange things so that the pieces are all the same size.

If we divide each piece of the top unit into 5 equal pieces and each piece of the bottom unit into 7 equal pieces, both units will be in 35 equal pieces. Now we can just count the total number shaded, as before.

Figure 2.17: \( \frac{3}{7} = \frac{15}{35} \) and \( \frac{2}{5} = \frac{14}{35} \)

This subdividing is the same thing we did before to discuss equivalent fractions. Here, we are finding equivalent fractions that have the same denominators because we already know
how to add those. Namely, we have
\[
\frac{3}{7} = \frac{3 \cdot 5}{7 \cdot 5} = \frac{15}{35}\quad \text{and}\quad \frac{2}{5} = \frac{2 \cdot 7}{5 \cdot 7} = \frac{14}{35}.
\]
Thus,
\[
\frac{3}{7} + \frac{2}{5} = \frac{15}{35} + \frac{14}{35} = \frac{29}{35}.
\]

\[\Box\]

**Example 2.4.9.** Compute \(\frac{4}{6} + \frac{8}{10}\). We know how to add fractions with the same denominator, so we will choose representatives for \(\frac{4}{6}\) and \(\frac{8}{10}\) that have the same denominator.

\[
\frac{4}{6} = \frac{4 \cdot 10}{6 \cdot 10} = \frac{40}{60}\quad \text{and}\quad \frac{8}{10} = \frac{8 \cdot 6}{10 \cdot 6} = \frac{48}{60}.
\]
Notice that the factor we chose in both cases was the denominator of the other fraction. **This will always work!**

Thus
\[
\frac{4}{6} + \frac{8}{10} = \frac{40}{60} + \frac{48}{60} = \frac{88}{60}.
\]
Our result will reduce to
\[
\frac{88}{60} = \frac{22 \cdot 4}{15 \cdot 4} = \frac{22}{15}.
\]

\[\Box\]

**Example 2.4.10.** In the previous example, we could have found a smaller common denominator. Since
\[
\frac{4}{6} = \frac{4 \cdot 5}{6 \cdot 5} = \frac{20}{30}\quad \text{and}\quad \frac{8}{10} = \frac{8 \cdot 3}{10 \cdot 3} = \frac{24}{30},
\]
we have
\[
\frac{4}{6} + \frac{8}{10} = \frac{20}{30} + \frac{24}{30} = \frac{20 + 24}{30} = \frac{44}{30}.
\]
which will also reduce to \(\frac{22}{15}\).
Often it is convenient to find a least common denominator as in the previous example, but it is never necessary. In fact, it is often easier to proceed as in Example 2.4.9, at least when you are first learning to add fractions.

*Example 2.4.11.*

\[
\frac{2}{5} + \frac{17}{15} = \frac{2 \cdot 15}{5 \cdot 15} + \frac{17 \cdot 5}{15 \cdot 5} = \frac{30 + 85}{75} = \frac{115}{75} = \frac{5 \cdot 23}{5 \cdot 15} = \frac{23}{15}.
\]

These principles apply even if you don’t know what the actual numbers in the fraction are.

*Example 2.4.12.*

\[
\frac{a}{2} + \frac{b}{5} = \frac{a \cdot 5}{2 \cdot 5} + \frac{b \cdot 2}{5 \cdot 2} = \frac{5a + 2b}{10}.
\]

*Example 2.4.13.*

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.
\]

The prior examples illustrate the following theorem. You should study the Fraction Addition Theorem until the pattern is clear to you.

**Theorem 5 (Fraction Addition).** Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be fractions. Then

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.
\]
This theorem allows us to add any fractions.

*Example 2.4.14.*

\[
\frac{4}{7} + \frac{5}{8} = \frac{4(8) + 7(5)}{7(8)} = \frac{32 + 35}{56} = \frac{67}{56}.
\]

The theorem automatically takes care of finding a common denominator.

□

*Example 2.4.15.*

\[
\frac{7}{-6} + \frac{12}{5} = \frac{7(5) + (-6)(12)}{(-6)(5)} = \frac{35 - 72}{-30} = \frac{-37}{-30} = \frac{37(-1)}{30(-1)} = \frac{37}{30}.
\]

□

*Example 2.4.16.* In this example, we need to convert 3 to the fraction \( \frac{3}{1} \) in order to use the Fraction Addition Theorem.

\[
3 + \frac{6}{13} = \frac{3}{1} + \frac{6}{13} = \frac{3 \cdot 13 + 6 \cdot 1}{1 \cdot 13} = \frac{45}{13}.
\]

□
Example 2.4.17.

\[
\frac{x}{y} + \frac{6x}{7} = \frac{x \cdot (7) + 6x \cdot y}{y \cdot (7)} = \frac{7x + 6xy}{7y}.
\]

Example 2.4.18.

\[
x + \frac{6x}{z} = \frac{x \cdot z + 6x \cdot z}{z \cdot z} = \frac{xz + 6xz}{z^2} = \frac{z \cdot (x + 6x)}{z \cdot z} = \frac{7x}{z}.
\]

This example shows that the Fraction Addition Theorem works even if we already have a common denominator.

Section 2.4 Exercises

Visually add the fractions represented by each model by first subdividing the diagram.

1.  
2.  
3.  
4.  
5.  
6.  

Compute as indicated and simplify if possible.
Compute as indicated and simplify if possible.

25. \( \frac{10}{x} + \frac{14}{x} \)

26. \( \frac{3y}{4x + 2} + \frac{5y}{4x + 2} \)

27. \( \frac{x + y}{6} + \frac{x - y}{6} \)

28. \( \frac{4x}{y - 3} - \frac{4x}{y - 3} \)

29. \( \frac{x}{(x+1)(x+2)} + \frac{1}{(x+1)(x+2)} \)

30. \( \frac{x}{x+1} + \frac{x + 1}{x} \)

31. \( \frac{4 + t}{4 - t} + \frac{1}{t + 2} \)

32. \( \frac{3}{8} + \frac{x + 4}{x + 2} \)

33. \( \frac{2x - 1}{3} - \frac{x}{x + 5} \)

34. \( \frac{3y + z}{5z} + \frac{y + 3z}{5y} \)

35. \( \frac{-12}{z} - \frac{11 + z}{-3z} \)

36. \( \frac{8z - 3x + 4}{7} = \frac{2x + 3}{5} \)

37. A board, known as a \( 2 \times 4 \), measures \( \frac{7}{2} \) inches wide by \( \frac{3}{2} \) inches deep. If two such boards are nailed together in an L-shape (so that the depth of one is nailed to the width of the other), what is the total width of the L?

38. James’ stock rose \( 5\frac{1}{8} \) on Monday and fell \( 2\frac{3}{8} \) on Tuesday. What was the net change in value?

39. Sarah installed baseboards in her house that were \( 3\frac{5}{8} \) inches tall. On top of the baseboards, she put molding that extended an additional \( 9\frac{1}{16} \) of an inch. What is the combined height of the baseboards and molding?

### 2.5 Operations on Fractions: Multiplication and Division

A “six-pack” of soda contains six cans. If we have 3 of the six-packs, how many cans do we have? We could add:

\[ 6 + 6 + 6 = 18, \]
but instead, we usually multiply:

\[ 3 \cdot 6 = 18. \]

Likewise, light bulbs often come in four-packs. If we have 8 of those, how many light bulbs do we have? Again, we multiply:

\[ 8 \cdot 4 = 32, \]

so we have 32 light bulbs.

Notice that the word “of” was in italics above. It is a word that often indicates that a multiplication is in order. It is reasonable to seek a definition of multiplication (for fractions) that will be consistent with this. That is,

\[ \frac{2}{3} \cdot \frac{6}{8} \]

should mean “\( \frac{2}{3} \) of \( \frac{6}{8} \),” and we define it so that it does.

**Example 2.5.1.** Suppose that we have \( \frac{6}{8} \) of a pie, as shown. What is \( \frac{2}{3} \) of the quantity that remains? When we say \( \frac{2}{3} \) of that quantity, we are indicating that we have a new unit; namely, the shaded portion. To find \( \frac{2}{3} \) of that, we must divide the shaded portion into 3 equal pieces, and take 2 of those. The heavy lines indicate the division of \( \frac{6}{8} \) into 3 equal pieces.

![Figure 2.18: \( \frac{2}{3} \) of \( \frac{6}{8} \) is \( \frac{4}{8} \).](image)

By counting the heavily shaded portions, we can see that \( \frac{2}{3} \) of \( \frac{6}{8} \) is \( \frac{4}{8} \).

The “pie model” for fractions is not very convenient for finding fractions of fractions. Instead, we will return to a rectangular model.

**Example 2.5.2.** What is \( \frac{4}{5} \) of \( \frac{3}{7} \)?

**Solution:** To determine this, we first need a diagram that illustrates \( \frac{3}{7} \).

![Figure 2.19: \( \frac{3}{7} \)](image)
The shaded portion is $\frac{3}{7}$, so we need to find $\frac{4}{5}$ of that. To accomplish this, we will divide the unit (the large rectangle) into 5 equal pieces. This will also subdivide each of the small shaded rectangles into 5 equal pieces.

![Figure 2.20: $\frac{3}{7}$ and the unit divided into 5 equal pieces](image)

Finally, we will shade 4 of the 5 horizontal rectangles. This causes some of the small squares to be shaded twice; they are shaded darker in the figure. The darkly shaded portion represents $\frac{4}{5}$ of $\frac{3}{7}$. Notice also the symmetry in the figure; $\frac{3}{7}$ of $\frac{4}{5}$ would look just the same, so these are equal quantities.

![Figure 2.21: $\frac{4}{5}$ of $\frac{3}{7}$](image)

If you just consider the portion that was originally shaded, $\frac{3}{7}$, you can see that that was subdivided into 5 equal pieces, and 4 of those were shaded more darkly. We can count that there are 12 darkly shaded squares, or we can multiply $3 \cdot 4 = 12$ since each of the 3 vertical rectangles contributed 4 squares to the darkly shaded area. Similarly, the entire rectangle was divided into 35 equal pieces since each of the 7 original vertical rectangles contributed 5 squares to the total: $7 \cdot 5 = 35$.

Overall, then, we have that $\frac{4}{5}$ of $\frac{3}{7}$ is $\frac{12}{35}$. In hindsight, we could have found this by multiplying numerators and denominators:

$$\frac{4}{5} \text{ of } \frac{3}{7} \text{ is } \frac{4 \cdot 3}{5 \cdot 7} = \frac{12}{35}.$$

The above example illustrates the following definition.

**Definition 2.5.1** (Fraction Multiplication). Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

We have chosen this definition so that multiplication corresponds to the word “of” in the same way it did for integers. Thus, $\frac{3}{4}$ of $\frac{5}{8}$ is

$$\frac{3}{4} \cdot \frac{5}{8} = \frac{3 \cdot 5}{4 \cdot 8} = \frac{15}{32}.$$
Example 2.5.3. What is $\frac{2}{5}$ of $\frac{3}{4}$?

Solution: We can apply our definition to find

$$\frac{2}{5} \cdot \frac{3}{4} = \frac{2 \cdot 3}{5 \cdot 4} = \frac{6}{20} = \frac{3}{10}.$$ 

Alternatively, we can consider the model again.

We begin with $\frac{3}{4}$, as shown.

![Figure 2.22: $\frac{3}{4}$](image)

To find $\frac{2}{5}$ of the shaded region, we must subdivide it into 5 equal pieces and shade 2 of those. In fact, we again subdivide the original unit into 5 equal pieces and shade 2 of those; this accomplishes the same thing, but also allows us to see how many of the resulting small squares it takes to fill the unit. The darkly shaded portion is the final result.

![Figure 2.23: $\frac{2}{5}$ of $\frac{3}{4}$](image)

Since each of the 4 vertical rectangles contributes 5 squares to the full unit, the unit is made up of $4 \cdot 5 = 20$ squares. Since each of the 3 shaded vertical rectangles contributes 2 to the darkly shaded region, there are $3 \cdot 2 = 6$ squares in there. Thus, we have that $\frac{2}{5}$ of $\frac{3}{4}$ is $\frac{6}{20} = \frac{3}{10}$, as before.

$\square$

Example 2.5.4.

$$\frac{7}{12} \cdot \frac{5}{3} = \frac{7 \cdot 5}{12 \cdot 3} = \frac{35}{36}.$$
The definition of multiplication applies to any fractions; the numerator and denominator need not be positive.

*Example 2.5.5.*

\[
\frac{-3}{2} \cdot \frac{5}{-9} = \frac{(-3)(5)}{2(-9)} = \frac{-15}{-18} = \frac{5(-3)}{6(-3)} = \frac{5}{6}.
\]

\[\square\]

**Definition 2.5.2.** If \(\frac{a}{b}\) is a nonzero fraction, then \(\frac{b}{a}\) is the **reciprocal** of \(\frac{a}{b}\). Likewise, \(\frac{a}{b}\) is the reciprocal of \(\frac{b}{a}\).

*Example 2.5.6.* The reciprocal of \(\frac{2}{5}\) is \(\frac{5}{2}\). Notice that

\[
\frac{2}{5} \cdot \frac{5}{2} = \frac{2 \cdot 5}{5 \cdot 2} = \frac{10}{10} = 1.
\]

\[\square\]

The result in the previous example is true in general: the product of reciprocal fractions is 1.

*Example 2.5.7.*

\[
\frac{7}{11} \cdot \frac{11}{7} = \frac{7 \cdot 11}{11 \cdot 7} = \frac{77}{77} = 1.
\]

\[\square\]
Example 2.5.8.

\[
\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = 1.
\]

Just as subtraction is defined in terms of addition, so is division defined in terms of multiplication. They are “inverse” operations, which is to say that division “undoes” multiplication.

Example 2.5.9. We know that \(4 \cdot 3 = 12\), which also means that \(12 \div 3 = 4\) and \(12 \div 4 = 3\).

The question now is, how do we divide by fractions? The idea of division is to split up a number into equal pieces that add up to that number. For example, when we divide 12 by 3, we are asking, how many groups of size 3 are there in 12? Similarly, when we divide \(\frac{3}{4}\) by \(\frac{1}{8}\), we are really asking, how many groups of size \(\frac{1}{8}\) are there in \(\frac{3}{4}\)?

Example 2.5.10. How many groups of size \(\frac{1}{8}\) are there in \(\frac{3}{4}\)?

**Solution:** The strip models below show \(\frac{3}{4}\) and \(\frac{1}{8}\).

![Figure 2.24: 3/4 and 1/8](image)

With these particular fractions, it is not hard to see that it will take 6 pieces of size \(\frac{1}{8}\) to make \(\frac{3}{4}\). Thus,

\[
\frac{3}{4} \div \frac{1}{8} = 6.
\]

Example 2.5.11. What is \(\frac{4}{5} \div \frac{2}{3}\)?

**Solution:** We have fraction strips for \(\frac{4}{5}\) and \(\frac{2}{3}\) below.
It is much more difficult to see how many pieces of size $\frac{2}{3}$ there are in $\frac{4}{5}$ because the sizes of the pieces are not the same. However, if we divide each piece of the $\frac{4}{5}$ model into 3 equal pieces and each piece of the $\frac{2}{3}$ model into 5 equal pieces, we will be able to compare them more readily. (This is similar to what we did with our strip model for addition.)

Now we can see that the $\frac{2}{3}$ has been broken into 10 equal pieces: each of the 2 original shaded pieces contributed 5 pieces when subdivided, and $2 \cdot 5 = 10$.

It takes 12 of those small pieces to make up $\frac{4}{5}$: each of the original 4 shaded pieces contributed 3 pieces when subdivided, and $4 \cdot 3 = 12$.

Therefore,

$$\frac{4}{5} \div \frac{2}{3} = \frac{4 \cdot 3}{2 \cdot 5} = \frac{12}{10}.$$ 

It is important to observe here that $\frac{12}{10}$ is not $\frac{12}{10}$ of the original unit, but of $\frac{2}{3}$. (That is, after all, what the question was!) We can check that we have done this correctly: $\frac{12}{10}$ of $\frac{2}{3}$ is

$$\frac{12 \cdot 2}{10 \cdot 3} = \frac{24}{30} = \frac{4 \cdot 6}{5 \cdot 6} = \frac{4}{5},$$

which is what we expect.
In the previous example, we saw that
\[
\frac{4}{5} \div \frac{2}{3} \text{ and } \frac{4}{5} \cdot \frac{3}{2}
\]
were the same quantity. In fact, this pattern always holds.

**Theorem 6 (Fraction Division).** Let \(\frac{a}{b}\) and \(\frac{c}{d}\) be fractions, with \(\frac{c}{d} \neq 0\). Then
\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.
\]

**Example 2.5.12.**
\[
\frac{2}{3} \div \frac{5}{7} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}.
\]
\[\square\]

**Example 2.5.13.** Recall that the fraction bar is another way of expressing division. Thus,
\[
\frac{\frac{3}{11}}{\frac{2}{5}} = \frac{3}{11} \div \frac{2}{5} = \frac{3 \cdot 5}{11 \cdot 2} = \frac{15}{22}.
\]
\[\square\]

**Example 2.5.14.**
\[
\frac{1}{\frac{3}{8}} = 1 \div \frac{3}{8} = 1 \cdot \frac{8}{3} = \frac{8}{3}
\]
Thus, \(\frac{1}{\frac{3}{8}}\) is just the reciprocal of \(\frac{3}{8}\).
Example 2.5.15.

\[
\frac{1}{a} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a}.
\]

Again, we just get that \(\frac{1}{a}\) is the reciprocal of \(\frac{a}{b}\).

Example 2.5.16. We can also think about division algebraically. What is \(\frac{7}{8} \div \frac{2}{5}\)? Let us temporarily call the answer \(\frac{a}{b}\) since we don’t know yet. Thus,

\[
\frac{7}{8} \div \frac{2}{5} = \frac{a}{b},
\]

which means the same thing as

\[
\frac{7}{8} = \frac{a}{b} \cdot \frac{2}{5}.
\]

We have seen that multiplying a fraction by its reciprocal results in a product of 1, so we will multiply both sides of this last equation by the reciprocal of \(\frac{2}{5}\).

\[
\frac{7}{8} \cdot \frac{5}{2} = \frac{a}{b} \cdot \frac{2}{5} \cdot \frac{5}{2} = \frac{a}{b} \cdot 1 = \frac{a}{b}.
\]

Notice that both

\[
\frac{7}{8} \div \frac{2}{5} \quad \text{and} \quad \frac{7}{8} \cdot \frac{5}{2}
\]

are equal to \(\frac{a}{b}\). Thus, instead of dividing by \(\frac{2}{5}\), we can find the same result by multiplying by the reciprocal of \(\frac{2}{5}\), which is \(\frac{5}{2}\). That’s what the theorem says to do.
Finally, we can see that
\[
\frac{7}{8} \div \frac{2}{5} = \frac{7}{8} \cdot \frac{5}{2}
\]
\[
= \frac{2 \cdot 7}{5 \cdot 8}
\]
\[
= \frac{14}{40}
\]
\[
= \frac{7}{20}.
\]
□

Example 2.5.17. Simplify \(\frac{2}{x} \div \frac{6}{5x}\).

Solution. Paying close attention to the symbols, we notice that we have a complex fraction, that is; we have a fraction whose numerator and denominator are also fractions. The implied grouping means that
\[
\frac{2}{x} \div \frac{6}{5x} = \frac{\frac{2}{x}}{\frac{6}{5x}}.
\]
\[
= \frac{2}{x} \cdot \frac{5x}{6}
\]
\[
= \frac{2 \cdot 5 \cdot x}{x \cdot 3}
\]
\[
= \frac{10}{3}.
\]
□

Section 2.5 Exercises

Determine the value of each quantity described.
1. \(\frac{2}{3}\) of \(\frac{2}{5}\).
2. \(\frac{4}{5}\) of \(\frac{1}{8}\).
3. \(\frac{2}{3}\) of \(\frac{5}{12}\).
4. \(\frac{1}{2}\) of \(\frac{1}{2}\).
5. \(\frac{4}{3}\) of \(\frac{7}{8}\).
6. \(\frac{9}{10}\) of 150.
7. 4 divided by \(\frac{1}{2}\).
8. 4 divided in half.
9. 6 divided by \(\frac{1}{3}\).
10. 10 divided by \(\frac{3}{4}\).

Find the reciprocal of each fraction.

11. \(\frac{2}{7}\)
12. \(\frac{6}{13}\)
13. \(\frac{2}{15}\)
14. \(\frac{41}{133}\)
15. \(\frac{21}{-16}\)
16. \(\frac{-71}{38}\)
17. \(\frac{14}{15}\)

Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

19. \(\frac{3}{4} \cdot \frac{2}{5}\)
20. \(\frac{1}{3} \cdot \frac{6}{5}\)
21. \(\frac{7}{8} \cdot \frac{8}{5}\)
22. \(\frac{3}{13} \cdot \frac{1}{14}\)
23. \(\frac{11}{7} \cdot \frac{9}{12}\)
24. \(\frac{45}{4} \cdot \frac{4}{15}\)
25. \(\frac{4}{5} \cdot 10\)
26. \(\frac{3}{32} \cdot 16\)
27. \(5 \cdot \frac{4}{5}\)
28. \(-7 \cdot \frac{12}{7}\)
29. \(15 \cdot \frac{7}{5}\)
30. \(\frac{4}{21} \cdot (-14)\)

Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

31. \(\frac{-10}{21} \cdot \frac{7}{-5}\)
32. \(\frac{x + 1}{4} \cdot \frac{3}{-(x + 1)}\)
33. \(\frac{3 \cdot x}{x} \cdot \frac{3}{3}\)
34. \(\frac{8z}{15} \cdot \frac{45}{4(z - 6)}\)
35. \(\frac{5}{2t + 3} \cdot \frac{2t}{5}\)
36. \(\frac{-5(x + 2)}{x + 1} \cdot \frac{4(x + 1)}{2x + 1}\)

Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

37. \(\frac{1}{4} \div \frac{2}{5}\)
38. \(\frac{2}{3} \div \frac{3}{4}\)
39. \(\frac{1}{5} \div \frac{3}{5}\)
40. \(\frac{7}{11} \div \frac{15}{11}\)
41. \(\frac{12}{x} \div \frac{9}{x}\)
42. \(\frac{4}{7} \div \frac{4}{7}\)
43. \(\frac{3}{8} \div \frac{8}{3}\)
44. \(\frac{4}{5} \div \frac{2}{9}\)
45. \(5 \div \frac{5}{2}\)
46. \(6 \div \frac{3}{8}\)
47. \(-\frac{3}{12} \div \frac{4}{9}\)
48. \(\frac{44}{16} \div \frac{3}{32}\)
49. \(\frac{15}{18} \div \frac{2}{3}\)
50. \(-\frac{15}{71} \div \frac{-15}{32}\)
51. \(\frac{31}{17} \div \frac{31}{19}\)
52. \(-\frac{23}{4} \div 8\)
Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

53. \[
\frac{4x + 15}{12 - 2x} \div -3 \div \frac{x(6 - x)}{x}
\]

54. \[
\frac{x}{x + 1} \div \frac{x}{5}
\]

55. \[
\frac{x(x + 3)}{4(2x - 1)} \div \frac{3(x + 3)}{x}
\]

56. \[
\frac{(x + 1)(x + 2)}{(x + 4)(x + 5)} \div \frac{(x + 1)}{(x + 4)}
\]

Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

57. \[
\frac{5}{x} + \frac{3}{x + 1}
\]

58. \[
\frac{8}{11} - \frac{-3}{11}
\]

59. \[
\frac{21}{17} \cdot \frac{17}{5}
\]

60. \[
15 - \frac{17}{3}
\]

61. \[
\frac{2}{3x - 18} \div \frac{4}{x - 6}
\]

62. \[
\frac{-32}{4} \div 8
\]

63. \[
-\frac{4}{3} + \frac{-10}{7}
\]

64. \[
\frac{3}{x} \cdot 4
\]

65. \[
\frac{10}{9} \cdot \frac{2x}{x + 2}
\]

66. \[
\frac{65}{16} + \frac{7}{3}
\]

67. \[
-\frac{64}{15} - \frac{12}{38}
\]
Compute as indicated, simplifying where possible. Be sure to include any implicit or explicit assumptions you may need.

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<td>( \frac{5}{11} )</td>
<td>72.</td>
<td>( \frac{\frac{14}{x+2}}{x+2} )</td>
</tr>
<tr>
<td>69.</td>
<td>( \frac{\frac{2}{3}}{x} )</td>
<td>71.</td>
<td>( \frac{\frac{5}{7}}{12} )</td>
<td>73.</td>
<td>( \frac{\frac{51}{18}}{90} )</td>
</tr>
</tbody>
</table>
| 74. | \( \frac{9}{3} \) | 76. | \( \frac{\frac{2x+1}{3}}{x} \)

78. Three-quarters of Andrews’ classmates belong to a campus club. If Andrew has 60 classmates, how many belong to a campus club? What percentage of his classmates belong to a campus club?

79. Marcus works at the bookstore two days out of every five all year through. If there are 365 days in a year, how many days in a year does Marcus work?

80. Marcy estimated that each person attending her party would eat 3 slices of pizza, where each pizza is cut into 8 slices. If Marcy has 24 guests, how many pizzas will she need?

81. Marcy can afford to buy 15 pizzas. If she expects each guest to eat \( \frac{1}{3} \) of a pizza, how many guests can she invite?

### 2.6 Applications of Fractions

Many phenomena in our experience can by described using ratios and proportions; this is what makes fractions so important. In this section we model some problems by using fraction concepts.

**Example 2.6.1.** In Tiffany’s class, \( \frac{3}{5} \) of the students are girls. If her class has 45 students in it, how many of them are girls?

**Solution:** We need to know \( \frac{3}{5} \) of 45; this is just the product of \( \frac{3}{5} \) and 45, as we have seen. Thus, there are

\[
\frac{3}{5} \cdot 45 = \frac{3 \cdot 45}{5} = \frac{3 \cdot 45}{5 \cdot 1} = \frac{135}{5} = 27
\]

girls in Tiffany’s class.

**Example 2.6.2.** A cookie recipe calls for \( 1 \frac{3}{4} \) cups of flour. How many cups are necessary to make 2 \( \frac{1}{2} \) batches?

**Solution:** First, notice that \( 1 \frac{3}{4} = \frac{7}{4} \) and \( 2 \frac{1}{2} = \frac{5}{2} \). Each batch requires \( \frac{7}{4} \) cups, and we need \( \frac{5}{2} \) of those; thus, we need

\[
\frac{5}{2} \cdot \frac{7}{4} = \frac{35}{8}
\]
cups, or \( 4 \frac{3}{8} \) cups. (35 divided by 8 is 4 with a remainder of 3.)
Example 2.6.3. Sarah earns $37,500 per year. If she receives a 5% raise, how much will she earn per year?

Solution: A 5% raise means that she will make her salary plus an additional 5% of her salary, so we need to compute 5% of $37,500. Remember that \( 5\% = \frac{5}{100} \), so we need to know \( \frac{5}{100} \) of $37,500. This is

\[
\frac{5}{100} \cdot \frac{37,500}{1} = \frac{187500}{100} = \frac{1875 \cdot 100}{1 \cdot 100} = 1875.
\]

This is the amount of her raise, so her income will be

\[
$37,500 + 1875 = $39,375.
\]

Definition 2.6.1. Two changing quantities are said to be proportional if their ratio is constant. If \( a \) and \( b \) are proportional, then \( a = kb \) for some constant \( k \), called the constant of proportionality.

Notice that if \( a \) is proportional to \( b \), then the ratio \( \frac{a}{b} = k \), for some constant \( k \).

Definition 2.6.2. A proportion is the equality of two ratios.

Example 2.6.4. If a person is travelling a constant speed, then the distance travelled is proportional to the time elapsed. That is, if \( d \) is the distance travelled and \( t \) is the time elapsed, then

\[ d = rt \]

for some constant \( r \) (the speed, or rate). For example, if Jim travels at a constant rate of 55 miles per hour, then

\[ d = 55t, \]

where \( d \) is Jim’s distance in miles and \( t \) is the time elapsed in hours.

Thus, after 1 hour, Jim has travelled \( d = 55(1) = 55 \) miles; after 2 hours, Jim has travelled \( d = 55(2) = 110 \) miles; and after 3.75 hours, Jim has travelled \( d = 55(3.75) = 206.25 \) miles.

Example 2.6.5. A certain car can travel 450 miles on 18 gallons of gasoline. At this rate, how many gallons of gasoline will be consumed on a 1,000 mile trip?

Solution 1. Let \( x \) denote the number of gallons of gasoline consumed on the 1,000 mile trip. Then the ratio of miles per gallon is constant so that

\[
\frac{450}{18} = \frac{1000}{x}.
\]

Using the Fraction Equivalence Theorem, we can solve this proportion as follows:
\[
\frac{450 \text{ miles}}{18 \text{ gallons}} = \frac{1000 \text{ miles}}{x \text{ gallons}} \quad \text{Given}
\]

\[
450 \cdot x = 18 \cdot 1000 \quad \text{Fraction Equivalence}
\]

\[
450x = 18,000 \quad \text{Multiplication}
\]

\[
x = \frac{18,000}{450} \quad \text{Divide by 450}
\]

\[
x = \frac{40}{1} \quad \text{Fraction Simplification}
\]

Thus, the car will consume 40 gallons of gasoline on a 1,000 mile trip.

**Solution 2.** Notice that the second trip is \(\frac{1000}{450} = 2.2\) times the distance of the first trip. Thus, the car will require \(2.2 \cdot 18 = 40\) gallons of gasoline on the second trip.

**Solution 3.** Let \(x\) denote the number of gallons of gasoline consumed on the 1,000 mile trip. We can write and solve the following proportion:

\[
\frac{1000}{450} = \frac{x}{18} \quad \text{Given}
\]

\[
1000 \cdot 18 = 450 \cdot x \quad \text{Fraction Equivalence}
\]

\[
18,000 = 450x \quad \text{Multiplication}
\]

\[
450x = 18,000 \quad \text{Symmetry of Equality}
\]

\[
x = \frac{18,000}{450} \quad \text{Divide by 450}
\]

\[
x = \frac{40}{1} \quad \text{Fraction Simplification}
\]

Thus, the car will consume 40 gallons of gasoline on a 1,000 mile trip.

**Solution 4.** Let \(x\) denote the number of gallons of gasoline consumed on the 1,000 mile trip. We can write and solve the following proportion:

\[
\frac{450}{1000} = \frac{18}{x} \quad \text{Given}
\]

\[
450 \cdot x = 1000 \cdot 18 \quad \text{Fraction Equivalence}
\]

\[
450x = 18,000 \quad \text{Multiplication}
\]

\[
x = \frac{18,000}{450} \quad \text{Divide by 450}
\]

\[
x = \frac{40}{1} \quad \text{Fraction Simplification}
\]

Thus, the car will consume 40 gallons of gasoline on a 1,000 mile trip.
Notice that we’ve presented four correct solutions to the above example. In general, a given proportion will yield several equivalent equalities:

\[
\frac{a}{b} = \frac{c}{d} \quad \text{Given Proportion} \\
\frac{ad}{bc} \quad \text{Fraction Equivalence} \\
\frac{a}{c} = \frac{b}{d} \quad \text{Fraction Equivalence} \\
\frac{c}{a} = \frac{d}{b} \quad \text{Fraction Equivalence} \\
\frac{b}{a} = \frac{d}{c} \quad \text{Fraction Equivalence}
\]

provided each of \(a, b, c,\) and \(d\) are nonzero so that each fraction makes sense.

Example 2.6.6. A pine tree casts an 11-foot shadow. At the same time, a woman, who is 5 feet 4 inches tall, casts a 16 inch shadow. Determine the height of the pine tree.

**Solution 1.** Let \(x\) denote the height of the pine tree (in feet). Converting the other measurements to feet we have that the woman is 5.3 feet tall and casts a 1.3 foot shadow.

![Figure 2.27: Shadows of tree and woman](image)

At a given time, the ratio of height to shadow length is constant; we also say that the triangles shown are similar. Therefore, we have the proportion

\[
\frac{x}{5.3} = \frac{11}{1.3} \quad \text{Given Proportion} \\
x \cdot 1.3 = 5.3 \cdot 11 \quad \text{Fraction Equivalence} \\
1.3x = 58.6 \quad \text{Multiplication} \\
x = 44 \quad \text{Divide by 1.3 and simplify}
\]

Thus, the pine tree is 44 feet tall.
Solution 2. Let $x$ denote the height of the pine tree (in feet). Converting the other measurements to (fractional) feet we have that the woman is $5\frac{1}{3} = \frac{16}{3}$ feet tall and casts a $1\frac{1}{3} = \frac{4}{3}$ foot shadow. Thus, we have the proportion

$$\frac{x}{\frac{16}{3}} = \frac{11}{\frac{4}{3}} \quad \text{Given Proportion}$$

$$x \cdot \frac{4}{3} = \frac{16}{3} \cdot 11 \quad \text{Fraction Equivalence}$$

$$\frac{4x}{3} = \frac{176}{3} \quad \text{Multiplication}$$

$$4x = 176 \quad \text{Multiply by 3}$$

$$x = 44 \quad \text{Divide by 4}$$

Thus, the pine tree is 44 feet tall.

Solution 3. Let $x$ denote the height of the pine tree (in feet). We also have the proportion

$$\frac{x}{11} = \frac{\frac{16}{3}}{\frac{4}{3}} \quad \text{Given Proportion}$$

$$\frac{x}{11} = \frac{16}{4} \quad \text{Fraction Equivalence}$$

$$\frac{x}{11} = 4 \quad \text{Division}$$

$$x = 4 \cdot 11 \quad \text{Multiply by 11}$$

$$x = 44 \quad \text{Multiplication}$$

Thus, the pine tree is 44 feet tall.

Section 2.6 Exercises

Solve each proportion.

1. $\frac{x}{3} = \frac{5}{6}$
2. $\frac{a}{5} = \frac{12}{20}$
3. $\frac{3}{x} = \frac{5}{8}$
4. $\frac{2}{x} = \frac{15}{16}$
5. $\frac{8}{7} = \frac{x}{4}$
6. $\frac{7}{12} = \frac{x}{15}$
7. $\frac{4}{9} = \frac{5}{z}$
8. $\frac{3}{5} = \frac{9}{w}$
9. $\frac{4}{7} = \frac{5}{2}$
10. $\frac{x}{40} = \frac{28}{135}$
11. $\frac{70}{15} = \frac{55}{x}$
12. $\frac{30}{24} = \frac{x}{15}$

13. A car consumes 7 gallons of gasoline on a 250-mile trip at 55 mph. How much gasoline does it consume on a 450-mile trip at the same speed?
14. The rate that water from a hose fills a swimming pool is proportional to the amount of time elapsed. If it takes 12 minutes for 150 gallons, how long will it take for 6000 gallons?

15. The radius of a circle is proportional to its circumference. If a circle with a circumference of 12 meters has a radius of approximately 1.90986 meters, what (approximately) is the radius of a circle with a circumference of 20 meters?

16. A building casts a 120-foot shadow at 9 AM. Also at 9 AM, a 50-foot flagpole casts a 30-foot shadow. How tall is the building?

17. The radius of a circle is proportional to its circumference. If a circle with a circumference of 12 meters has a radius of approximately 1.90986 meters, what (approximately) is the radius of a circle with a circumference of 20 meters?

18. The force exerted by a spring is proportional to how far it is stretched or compressed. If the force is 45 pounds when stretched 8 inches, what force will it exert if it is only stretched 3 inches?

19. A spring is compressed 3 cm by a force of 5 Newtons. How far will a force of 32 Newtons compress the spring?

20. If 3 pizzas can feed 8 children, how many children can 12 pizzas feed?

21. The thickness of a stack of paper is proportional to the number of sheets in the stack. If a stack of 500 sheets is 2 inches thick, how thick is a stack of 1250 sheets?

22. The amount of paint needed to paint a room is proportional to the area of the walls. If a room with 1000 square feet of walls space requires 2.5 gallons of paint, how many gallons of paint are needed for a room with 1750 square feet of wall space?

23. Scale models are models of objects whose measurements are in proportion to the actual objects. A scale model of a jet is 15 inches long. If the constant of proportionality is 80, how long is the actual jet?

24. A scale model of a race car has a wheel that is 1.5 inches in diameter; the wheel of the actual race car has a diameter of 24 inches. If the model is 12 inches long, how long is the actual race car?

**Similar triangles** are triangles whose corresponding sides are in proportion, as in Example 2.6.6. A theorem from Geometry states that triangles are similar if two angles of one triangle are the same as two angles of the other triangle. (In Example 2.6.6, both triangles had a right angle, and they shared the angle at A.) For the following problems, (a) verify that the triangles in each pair are similar and (b) use a proportion to find the missing length.
2.7 Rational Expressions

The preceding sections have provided the background for the concept of fractions and explained why we treat fractions the way we do. Now we will introduce the idea of a rational expression, which is a generalization of the idea of a fraction.

Definition 2.7.1. A rational expression is a fraction that may (or may not) include a variable (or variables) in the numerator and/or denominator.

Example 2.7.1. The following are examples of rational expressions.

\[
\frac{3}{8}, \quad \frac{x+1}{5}, \quad \frac{6}{xy+8x-3}, \quad \frac{(x+1)(x-4)}{(x+2)(4x+3)}, \quad \frac{x+y+z-w}{4x-w}.
\]

Since rational expressions are defined in terms of fractions, we will apply the same definitions for operations, equivalence, and simplification as we did for fractions. Notice that this is much more abstract now: it doesn’t mean much to say that we are dividing a unit up into \(x+5\) equal pieces, so the rational expression

\[
\frac{x-3}{x+5}
\]

must be taken as just an object in its own right. The important thing to remember is that it is analogous to a regular fraction, and it will behave the same way.

Since rational expressions are modelled after fractions, they are simplified in the same way as fractions.

Theorem 7. The rational expression

\[
\frac{pr}{qr}
\]

is equal to the rational expression \(\frac{p}{q}\).

Example 2.7.2.

\[
\frac{(3x+2)(5x-7)}{(3x-2)(5x-7)} = \frac{3x+2}{3x-2}, \quad x \neq \frac{7}{5}
\]

The numerator and denominator share the factor of \(5x-7\). When this common factor is “cancelled,” the assumption that \(x \neq \frac{7}{5}\) must be explicitly made. Note that the assumption that \(x \neq \frac{2}{5}\) is implicit in both expressions; we can see that we must make this assumption by considering the denominator.
Example 2.7.3.

\[
x(x + 2) \quad x(x - 3) = \frac{x + 2}{x - 3}, \quad x \neq 0
\]

Example 2.7.4.

\[
\frac{x + 2}{x + 4}
\]

cannot be simplified since the numerator and denominator do not have a factor in common. That \(x \neq -4\) is implicit for the expression to be defined.

Example 2.7.5.

\[
\frac{x - 5}{4x - 20} = \frac{x - 5}{4(x - 5)} = \frac{1(x - 5)}{4(x - 5)} = \frac{1}{4}, \quad x \neq 5.
\]

Notice that we had to apply the distributive law from Chapter 1 to the numerator before we could simplify. The distributive law will be covered in detail in Chapter 4.

Example 2.7.6.

\[
\frac{x(x + 2) - 4(x + 2)}{(x - 4)(3x + 5)} = \frac{(x - 4)(x + 2)}{(x - 4)(3x + 5)} = \frac{x + 2}{3x + 5}, \quad x \neq 4
\]

Example 2.7.7.

\[
\frac{4(5 - x)}{12(5 + x)} = \frac{4(5 - x)}{4 \cdot 3(5 + x)} = \frac{5 - x}{3(5 + x)}
\]

All of the operations on rational expressions are defined by analogy with operations on fractions since they are set up in terms of fractions.

**Definition 2.7.2.** Let \(\frac{p}{r}\) and \(\frac{q}{r}\) be rational expressions with a common denominator. Then

\[
\frac{p}{r} + \frac{q}{r} = \frac{p + q}{r}
\]

and

\[
\frac{p}{r} - \frac{q}{r} = \frac{p - q}{r}.
\]
**Definition 2.7.3.** Let $\frac{p}{q}$ and $\frac{r}{s}$ be rational expressions. Then

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$$

and

$$\frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}$$

**Example 2.7.8.** Since $\frac{3}{x}$ and $\frac{4x}{x}$ have the same denominator, we may use the first definition to get

$$\frac{3}{x} + \frac{4x}{x} = \frac{3 + 4x}{x}.$$ 

\[\square\]

**Example 2.7.9.** Since $\frac{x + 2}{x}$ and $\frac{5 - x}{x}$ have the same denominator, we may use the first definition to get

$$\frac{x + 2}{x} + \frac{5 - x}{x} = \frac{(x + 2) + (5 - x)}{x} = \frac{7}{x}$$

since the $x$ and the $-x$ cancel each other. Note that the only necessary assumption is the *implicit* assumption that $x \neq 0$.

Notice that $(x + 2)$ and $(5 - x)$ were each grouped in the middle expression. Remember, the fraction bar acts as a grouping symbol, and we need to keep that in mind when we compute with rational expressions.

\[\square\]

**Example 2.7.10.**

$$\frac{x + 2}{x + 1} - \frac{x + 4}{x + 1} = \frac{(x + 2) - (x + 4)}{x + 1} = \frac{x + 2 - x - 4}{x + 1} = \frac{-2}{x + 1}.$$ 

Again, the fraction bar indicates a grouping, so the whole numerator $(x + 4)$ must be subtracted. That means we have to subtract the $x$ and the 4.

\[\square\]

**Example 2.7.11.** Since $\frac{-1}{x}$ and $\frac{4}{x+2}$ have different denominators, we will use the second definition.

$$\frac{-1}{x} + \frac{4}{x+2} = \frac{(-1)(x + 2) + (4)(x)}{x(x + 2)} = \frac{(-1)(x) + (-1)(2) + 4x}{x(x + 2)},$$

where in the last step we have again used the distributive law from Chapter 1. This then becomes

$$\frac{-x - 2 + 4x}{x(x + 2)} = \frac{3x - 2}{x(x + 2)},$$

We again used the distributive law to simplify $-x - 2 + 4x$ to $3x - 2$. 
Of course, we can still add by finding a common denominator:

\[
\frac{-1}{x} = \frac{(-1)(x + 2)}{x(x + 2)} = \frac{-x - 2}{x(x + 2)} \quad \text{and} \quad \frac{4}{x + 2} = \frac{x \cdot 4}{x(x + 2)}.
\]

Thus,

\[
\frac{-1}{x} + \frac{4}{x + 2} = \frac{-x - 2 + 4x}{x(x + 2)} = \frac{3x - 2}{x(x + 2)},
\]

and we arrive at the same result as before.

\(\square\)

**Example 2.7.12.**

\[
\frac{1}{x + 3} - \frac{5}{x - 2} = \frac{1(x - 2) - 5(x + 3)}{(x + 3)(x - 2)} = \frac{x - 2 - 5x - 15}{(x + 3)(x - 2)} = \frac{-4x - 17}{(x + 3)(x - 2)}.
\]

\(\square\)

**Example 2.7.13.**

\[
3x + 1 + \frac{4x - 7}{6} = \frac{3x + 1}{1} + \frac{4x - 7}{6} = \frac{(3x + 1)(6) + 1(4x - 7)}{(1)(6)} = \frac{18x + 6 + 4x - 7}{6} = \frac{22x - 1}{6}.
\]

\(\square\)

**Definition 2.7.4.** If \(\frac{p}{q}\) and \(\frac{r}{s}\) are rational expressions, then

\[
\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.
\]
Definition 2.7.5. If $\frac{p}{q}$ and $\frac{r}{s}$ are rational expressions and $\frac{r}{s} \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}.$$

These operations are again defined in terms of the same operations on fractions.

Example 2.7.14.

$$\frac{x}{x + 2} \cdot \frac{x + 2}{x - 1} = \frac{x(x + 2)}{(x + 2)(x - 1)} = \frac{x}{x - 1}, \quad x \neq -2$$

Since $x \neq 1$ is implicit in all of the above expressions, it need not be specified. That $x \neq -2$, on the other hand, must be made explicit for the equality to hold.

Example 2.7.15.

$$\frac{(x + 3)(x + 2)}{(x - 2)(3x + 4)} \cdot \frac{(3x + 4)(x - 1)}{(x + 3)(x + 2)} = \frac{[(x + 3)(x + 2)][(3x + 4)(x - 1)]}{[(x - 2)(3x + 4)][(x + 3)(x + 2)]}$$

$$= \frac{x + 1}{x - 2}, \quad x \neq -\frac{4}{3}, x \neq -3, x \neq -2.$$

Notice, again, that the implicit assumption $x \neq 2$ is implied in all of the above expressions and need not be specified.

Example 2.7.16.

$$\frac{x(x - 3)}{x + 2} \div \frac{x(x + 1)}{(2x - 3)(x + 2)} = \frac{x(x - 3)}{x + 2} \cdot \frac{(2x - 3)(x + 2)}{x(x + 1)}$$

$$= \frac{x(x - 3)(2x - 3)(x + 2)}{(x + 2)(x)(x + 1)}$$

$$= \frac{(x - 3)(2x - 3)}{x + 1}, \quad x \neq -2, x \neq 0.$$

Example 2.7.17.

$$\frac{(x + 1)(x - 3)}{3x + 2} \cdot \frac{(3x + 2)(2x + 6)}{x + 1} = \frac{(x + 1)(x - 3)(3x + 2)(2x + 6)}{(3x + 2)(x + 1)}$$

$$= (x - 3)(2x + 6).$$
Example 2.7.18.

\[
\frac{(3x+4)(x-5)}{(2x+1)(x+2)} = \frac{(3x + 4)(x - 5)}{(2x + 1)(x + 2)} \div \frac{(3x + 4)x}{(2x + 1)(x - 2)}
\]

\[
= \frac{(3x + 4)(x - 5)}{(2x + 1)(x + 2)} \cdot \frac{(2x + 1)(x - 2)}{(3x + 4)x}
\]

\[
= \frac{(3x + 4)(x - 5)(2x + 1)(x - 2)}{(2x + 1)(x + 2)(3x + 4)x}
\]

\[
= \frac{(x - 5)(x - 2)}{(x + 2)x}.
\]

\[\square\]

Example 2.7.19.

\[
(x + 5) \cdot \frac{x - 2}{3x + 1} = \frac{x + 5}{1} \cdot \frac{x - 2}{3x + 1}
\]

\[
= \frac{(x + 5)(x - 2)}{(1)(3x + 1)}
\]

\[
= \frac{(x + 5)(x - 2)}{3x + 1}.
\]

\[\square\]

Example 2.7.20.

\[
(3x - 17) \cdot \frac{4x + 9}{3x - 17} = \frac{3x - 17}{1} \cdot \frac{4x + 9}{3x - 17}
\]

\[
= \frac{(3x - 17)(4x + 9)}{(1)(3x - 17)}
\]

\[
= \frac{4x + 9}{1}
\]

\[
= 4x + 9
\]

\[\square\]
Example 2.7.21.

\[
(4x + 2) \div \frac{3x - 8}{2x + 1} = \frac{4x + 2}{1} \div \frac{3x - 8}{2x + 1} \\
= \frac{4x + 1}{2} \cdot \frac{2x + 1}{3x - 8} \\
= \frac{(4x + 1)(2x + 1)}{2(3x - 8)}.
\]

\[\Box\]

Section 2.7 Exercises

Simplify each rational expression, if possible.

1. \(\frac{x(x - 2)}{x(x + 2)}\)
2. \(\frac{x + 2}{3(x + 2)}\)
3. \(\frac{3x - 5}{9x - 15}\)
4. \(\frac{(x + 2)(x + 1)}{5(x + 1)}\)
5. \(\frac{4x(2x - 1) - 3(2x - 1)}{2x - 1}\)
6. \(\frac{(x + 1)(x + 2)(x + 3)}{(x + 2)(x + 3)(x + 1)}\)
7. \(\frac{-7x}{x(4 - 3x)}\)
8. \(\frac{t(t + 6)}{t(t + 6) - 5(t + 6)}\)
9. \(\frac{x + 4}{x - 4}\)
10. \(\frac{(2x + 1)x}{8x(6x + 3)}\)
11. \(\frac{4x}{x(7x - 1)}\)

Compute as indicated and simplify. Be sure to state any assumptions you may need.

13. \(\frac{3}{4x} + \frac{7}{4x}\)
14. \(\frac{5 - x}{x + 1} - \frac{x + 2}{x + 1}\)
15. \(\frac{8t + 2}{5} + \frac{3}{5}\)
16. \(\frac{9z - 5}{(z + 2)(z + 3)} - \frac{3z - 23}{(z + 2)(z + 3)}\)
17. \(\frac{x}{x - 2} \cdot \frac{x - 2}{x + 1}\)
18. \(\frac{x(x + 1)}{3x(2x - 2)} \cdot \frac{x - 1}{(x + 1)(4x - 1)}\)
19. \(\frac{5x + 2}{3x} \div \frac{5x + 2}{x - 3}\)
20. \(\frac{4x + 2}{(x - 3)(x - 5)} \div \frac{2x + 1}{x - 3}\)
21. \(\frac{8x - 12}{2x - \frac{4}{3}} \cdot \frac{x - \frac{2}{3}}{(2x - 3)(x + 1)}\)
22. \(\frac{4}{3x} + \frac{5}{2x - 7}\)
23. \(\frac{(5x + 24)(3x - 8)}{x(x - 3)} \cdot \frac{(21x - 4)x}{(3x - 8)(x + 2)}\)
24. \(\frac{x}{4} - \frac{5x + 2}{12}\)
25. \(\frac{4x}{6x} + \frac{2x - 14}{3x}\)

26. \(\frac{50x}{35(11x - 3)} \div \frac{2x}{4x + 1}\)

27. \(\frac{6}{x} - \frac{2}{12x + 15}\)

28. \(\frac{t(8t + 1)}{t + 8} \div \frac{(8t + 1)(15t - 2)}{6}\)

29. \(\frac{t + 5}{8} - \frac{24t - 5}{12}\)

30. \(\frac{yz(x + 3)}{(y - 1)(4z + 7)} \cdot \frac{(4z + 7)x}{y(x + 3)(z - 2)}\)

31. \(\frac{8x - 11}{(x + 3)(5x - 1)} + \frac{12}{(x + 3)}\)

32. \(\frac{3x + 1}{4x} - \frac{2x - 2}{8x}\)

33. \(\frac{15t(u + 1)}{20u(t + 1)} \div \frac{12t}{u}\)

34. \(\frac{28x + 14}{(x - 5)(3x + 13)} \cdot \frac{(x - 5)(3x + 13)}{2x + 1}\)

35. \(-12\frac{2x - 9}{11x + 3}\)

36. \(\frac{14}{8x + 6} \div \frac{x - 7}{4x + 3}\)

37. \(3x - 6 + \frac{4x - 9}{12}\)

38. \(\frac{3x + 2}{x - 8} \div (8x - 2)\)

39. \(\frac{(4x + 5) \div 3x + 5}{7x - 1}\)

40. \((6x - 11) \cdot \frac{2x + 4}{x - 5}\)

Compute as indicated and simplify. Be sure to state any assumptions you may need.

41. \(\frac{4x}{x + 5} \div \frac{12x}{x - 5}\)

42. \(\frac{3x}{(x + 2)(3x - 6)} \div \frac{9}{x - 2}\)

43. \(\frac{51x - 17}{x(2x + 1)} \div \frac{12x - 1}{x(x - 14)}\)

44. \(\frac{8x(8x - 5)}{12x + 2} \div \frac{4x}{6x + 2}\)

45. \(\frac{(4x - 3)(3x - 1)}{4x - 1} \div \frac{12x - 9}{(x + 4)(3x - 2)}\)

46. \(\frac{(t + 15)(6t - 5)}{(t - 2)(t + 4)} \div \frac{t + 15}{(t - 2)t}\)
Chapter 3

Exponents

3.1 Introduction to Exponents

In mathematics, there are a number of shorthand notations that simplify what it takes to convey meaning. For example, instead of saying

\[ 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 \]

we usually just write

\[ 4 \cdot 10, \]

for “4 times 10”. Thus, multiplication can be thought of as just repeated addition: we have ten terms, each of which is a 4, so we write \( 4 \cdot 10 \).

How could we express

\[ 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \]

in a more convenient form? We can’t write \( 4 \cdot 10 \) because that has already been taken; it’s what we use for repeated addition. Instead, to indicate a repeated multiplication, we use what are called exponents. For the above product of ten 4’s, we write

\[ 4^{10} \]

The 10 is called the exponent and the 4 is called the base. The exponent indicates the number of factors of the base that appear: because we had 10 factors, each of which was a 4, we wrote \( 4^{10} \).

**Definition 3.1.1.** Let \( a \) be a real number, and let \( n \) be a natural number. The symbol \( a^n \) is defined by

\[ a^n = a \cdot a \cdot a \cdot \ldots \cdot a. \]

The symbol \( a^n \) is pronounced “\( a \) to the \( n \)th power” or simply “\( a \) to the \( n \)”.” The number \( a \) is called the base and \( n \) is the exponent or power.

**Example 3.1.1.** Since the exponent tells us the number of factors of the base we have, if we see \( 4^1 \), it must just be the same as 4. In fact, \( 6^1 = 6, 48^1 = 48 \), and, in general,

\[ a^1 = a \]

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for any real number $a$. This goes both ways, too: if you see $11$, you can think of it as $11^1$ if that is helpful. (We will see as we progress that sometimes it is helpful!)

$\Box$

**Example 3.1.2.** When $n$ is 2 and $a$ is 5, we have

$$a^n = 5^2$$
$$= 5 \cdot 5$$
$$= 25.$$  

When the exponent is two, it is often pronounced as “squared” instead of “second power.” Thus, $5^2$ can be pronounced either “5 to the second power” or “5 squared.” Note that a 5 by 5 square has area of 25 units; hence, the terminology 25 “square” units.

$\Box$

**Example 3.1.3.** When $n$ is 5 and $a$ is 2, we have

$$a^n = 2^5$$
$$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2,$$

which can be simplified to 32.

$\Box$

**Example 3.1.4.** When $n$ is 2 and $a$ is $-10$, we have

$$(-10)^2 = (-10)(-10)$$
$$= 100.$$  

Notice that this is positive even though $-10$ is negative.

$\Box$

**Example 3.1.5.** When $n = 3$ and $a = 7$, we have $a^n = 7^3 = 7 \cdot 7 \cdot 7$, which can be simplified to 343. When the exponent is three, it is often pronounced as “cubed” instead of “third power.” Thus, $7^3$ can be pronounced either “7 to the third power” or “7 cubed.” Note that a 7 by 7 by 7 cube has volume of 343 units; hence, the terminology 343 “cubic” units.

$\Box$

**Example 3.1.6.** If $n = 4$ and $a = -5$, we have $a^n = (-5)^4 = (-5)(-5)(-5)(-5) = 625$. Notice that the minus sign is part of the number $-5$, so it must appear in each of the four factors.

$\Box$
Example 3.1.7. Suppose that \( n = 4 \) and \( a = xy \). Here, the \( x \) and the \( y \) represent unknown real numbers. We have

\[
a^n = (xy)^4
\]
\[
= (xy)(xy)(xy)(xy)
\]
\[
= (x \cdot x \cdot x)(y \cdot y \cdot y)
\]
\[
= x^4y^4.
\]

We had to use some of the properties of real numbers here; specifically, we needed to use that multiplication is both associative and commutative to put all of the \( x \)'s together and all of the \( y \)'s together.

Example 3.1.8. Suppose that \( n = 3 \) and \( a = \frac{2}{5} \). Then

\[
\left( \frac{2}{5} \right)^3 = \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5}
\]
\[
= \frac{2 \cdot 2 \cdot 2}{5 \cdot 5 \cdot 5}
\]
\[
= \frac{2^3}{5^3}
\]
\[
= \frac{8}{125}.
\]

Example 3.1.9. Simplify \( \frac{2^3}{2^5} \).

Solution:

\[
\frac{2^3}{2^5} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}
\]
\[
= \frac{1}{(2 \cdot 2)(2 \cdot 2)}
\]
\[
= \frac{1}{2 \cdot 2}
\]
\[
= \frac{1}{4}.
\]

Example 3.1.10. What should \( 2 \cdot 6^3 \) mean? We know that \( 6^3 = 6 \cdot 6 \cdot 6 \), so \( 2 \cdot 6^3 \) will mean

\[
2 \cdot 6 \cdot 6 \cdot 6 = 432.
\]

We have the three factors of 6 indicated, but only one factor of 2. On the other hand,

\[
(2 \cdot 6)^3 = 12^3 = 12 \cdot 12 \cdot 12 = 1728.
\]

The parentheses make a difference! That is, when we write \( 2 \cdot 6^3 \), we have the convention that parentheses are understood around the \( 6^3 \): \( 2 \cdot (6^3) \).
The previous example illustrates an important point: in terms of the order of operations, exponents receive a higher priority than multiplication or division.

**Example 3.1.11.** What is $-2^4$? We must again appeal to convention here. The expression $-2^4$ is similar in nature to $2 \cdot 6^3$ in that it can be written as $-1 \cdot 2^4$. This means that we have

$$-1 \cdot 2^4 = -1(2)(2)(2) = -16.$$  

There is only the one minus sign, but four 2’s. 

On the other hand,

$$( -2)^4 = (-2)(-2)(-2)(-2) = 16,$$

so the two expressions $-2^4$ and $(-2)^4$ mean different things. Watch out! Every symbol has a purpose; in particular, parentheses can change the meaning of a mathematical expression.

It is worth observing the following fact about even exponents.

**Theorem 8.** If $a$ is a base and $n$ is any even exponent, then

$$a^n \geq 0.$$  

This is reasonable since an even exponent of $n$ means that we have an even number of factors, all of which are the same. Thus, if we have a negative base, we end up with a product of an even number of negative numbers multiplied together, so the product is positive.

We continue this section with a brief discussion of polynomial expressions.

**Definition 3.1.2.** A polynomial in $x$ is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers, called **coefficients**, $n$ is a whole number, and $x$ is a variable. The **degree** of the term $a_ix^i$ is $i$, and the degree of the polynomial is the highest degree that appears ($n$, if $a_n \neq 0$).

We refer to the term $a_ix^i$ as the $x^i$ term or the degree-$i$ term. The degree-0 term is also called the **constant term** (since it has no variables). The highest degree term is called the **leading term** and its coefficient is the **leading coefficient**.

**Example 3.1.12.** $4x^3 + 7x - 5$ is a polynomial in $x$. The degrees and coefficients of each term are in the table below.

<table>
<thead>
<tr>
<th>Term</th>
<th>Degree</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$x^2$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$x$</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>constant</td>
<td>0</td>
<td>$-5$</td>
</tr>
</tbody>
</table>
The constant term is $-5$, the leading term is $4x^3$, the leading coefficient is $4$, and the degree of the polynomial is $3$. Notice that since there is no $x^2$ term, we can think of the $x^2$ term as having a coefficient of $0$.

\[ 4x^2 + \frac{1}{x} \] is not a polynomial in $x$ since it does not have the form specified in the definition.

$-7t^2 + 4t^6 - 11t^5 + t^3$ is a polynomial in $t$. The table below summarizes the information for this polynomial. To make it easier to gather all of the information, we will rewrite the polynomial in descending order of the exponents:

\[
4t^6 - 11t^5 + t^3 - 7t^2.
\]

<table>
<thead>
<tr>
<th>Term</th>
<th>Degree</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^6$</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$t^5$</td>
<td>5</td>
<td>$-11$</td>
</tr>
<tr>
<td>$t^4$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$t^3$</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$t^2$</td>
<td>2</td>
<td>$-7$</td>
</tr>
<tr>
<td>$t^1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>constant</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Notice that the coefficient of $t^3$ is $1$ since $t^3 = 1 \cdot t^3$. The degree of the polynomial is $6$, its leading coefficient is $4$, and its constant term is $0$.

In polynomial expressions, terms of the same degree are referred to as like terms.

In the polynomial expression

\[ 5x^2 - 7x + 13x^2 + 11, \]

the terms $5x^2$ and $13x^2$ are like terms; $-7x$ and $11$ are not like terms.

Let's briefly discuss one more use of exponents. In arithmetic, we learn to (prime-)factor natural numbers; for example,

\[ 72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3. \]

This can be expressed much more compactly as

\[ 72 = 2^3 \cdot 3^2. \]

Such a factorization, with the prime factors listed in increasing order, is called the prime-power representation of the number.
Example 3.1.16. Find the prime-power representation of 540.

Solution: We find the prime-power representation by finding one factor at a time.

\[
540 = 2 \cdot 270 \\
= 2 \cdot 2 \cdot 135 \\
= 2 \cdot 2 \cdot 3 \cdot 45 \\
= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 15 \\
= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \\
= 2^2 \cdot 3^3 \cdot 5.
\]

Section 3.1 Exercises

Express each verbal description symbolically.

1. Two to the fourth power.
2. The sum of three squared and two cubed.
3. The cube of the sum of two and three.
4. Twelve to the fifth.

Use Definition 3.1.1 to expand and evaluate each expression.

5. \(2^5\)  
13. \((-5)^4\)  
21. \((3^2)^3\)

6. \(6^3\)  
14. \(a^3\)  
22. \((\frac{3}{4})^2\)

7. \(2.5^3\)  
15. \((2a)^3\)  
23. \((\frac{4}{7})^3\)

8. \((-4)^3\)  
16. \(2a^3\)  
24. \((x^2)^4\)

9. \(-4^3\)  
17. \((3^4)(3^2)\)  
25. \((3x)^2\)

10. \((-2 + 1)^5\)  
18. \([(-5)^2]^3\)  
26. \((-\frac{3}{5})^3\)

11. \(4 \cdot 3^6\)  
19. \((2^5)(2^7)\)  
27. \((\frac{4}{3})^4\)

12. \(-5^4\)  
20. \((4^3)(5^3)\)  
28. \((3 + 4)^2\)

Identify the expressions below that are polynomials and those that are not.
Identify the degree, leading term, leading coefficient, and constant term of each polynomial.

35. $4x^3 + 5x - 7x^2$
36. $-\frac{3}{5}x^6 + 2x^9 - 1$
37. $4x^2 + 5x - 1$
38. $-x + 2$

Compute as indicated and simplify where possible.

47. $\frac{3x}{5} \cdot \frac{8x}{x+1}$
48. $\frac{3x}{x+1} - \frac{x}{2x-3}$

53. Identify which terms are like terms.

(a) $4x^5$
(b) $-3x^2$
(c) $15x^5$
(d) $5x^2$
(e) $\frac{3}{4}x^5$

(f) $\frac{2}{3}x^3$
(g) $-\frac{5}{8}x^4$
(h) $-15x^8$
(i) $31x^3$
(j) $-7x$

(k) $21x^6$
(l) $-5.9x^7$
(m) $36x^4$
(n) $5x^7$
(o) $11x^3$

(p) $-\frac{7}{2}x^8$
(q) $17x^6$
(r) $\frac{x^8}{3}$
(s) $\frac{64x^5}{7}$
(t) $-x$

Find the prime-power representation of each number.

54. $45$
55. $24$
56. $120$
57. $48$
3.2 Properties of Exponents

Notice that
\[
\begin{align*}
1^1 &= 1 \\
1^2 &= 1 \cdot 1 = 1 \\
1^3 &= 1 \cdot 1 \cdot 1 = 1 \\
1^4 &= 1 \cdot 1 \cdot 1 \cdot 1 = 1.
\end{align*}
\]

In fact, every power of 1 equals 1 since a power of 1 is really just a product of several 1’s.

**Theorem 9.** For any natural number \( n \), \( 1^n = 1 \).

Consider the following examples.

**Example 3.2.1.**

\[
(2^3) \cdot (2^4) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7.
\]

**Example 3.2.2.**

\[
(-3)^2(-3)^3 = [(-3)(-3)] \cdot [(-3)(-3)(-3)] = (-3)(-3)(-3)(-3)(-3) = (-3)^5.
\]

**Example 3.2.3.**

\[
x^4 \cdot x^1 = x^4 \cdot x = (x \cdot x \cdot x \cdot x) \cdot x = x \cdot x \cdot x \cdot x \cdot x = x^5.
\]

**Example 3.2.4.**

\[
a^m \cdot a^n = \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ factors}} \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} = \underbrace{a \cdot a \cdot a \cdots a}_{m + n \text{ factors}} = a^{m+n}.
\]
In each of the examples we just saw, we multiplied two powers of the same base \( (a) \) together. The result was that we acquired factors of the base from each power, and the exponent at the end was the sum of the original exponents. The theorem below summarizes this.

**Theorem 10 (Multiplication of Like Bases).** Let \( a \) be a real number and let \( m \) and \( n \) be natural numbers. Then

\[
a^m a^n = a^{m+n}.
\]

This theorem may be stated as follows: “To multiply like bases, add their exponents.” Here, “like bases” means that the bases are the same.

*Example 3.2.5.* \( 5^3 \cdot 5^4 = 5^{3+4} = 5^7 \). If we look at this in terms of the definition of exponents, we see that

\[
\begin{align*}
5^3 \cdot 5^4 &= (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \\
&= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \\
&= 5^7,
\end{align*}
\]

which agrees with the result we got by using the theorem.

*Example 3.2.6.*

\[
(-2)^4 \cdot (-2)^{12} = (-2)^{4+12} = (-2)^{16}.
\]

The same ideas apply to fractions and variables, as well as products with more than two factors.

*Example 3.2.7.*

\[
\frac{3^2}{5^3} \cdot \frac{3^7}{5^2} = \frac{3^2 \cdot 3^7}{5^3 \cdot 5^2} = \frac{3^{2+7}}{5^{3+2}} = \frac{3^9}{5^5}.
\]
Example 3.2.8.

\[
\left(\frac{5}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^8 = \left(\frac{5}{6}\right)^{4+8} = \left(\frac{5}{6}\right)^{12}
\]

We will see later how we can further simplify \( \left(\frac{5}{6}\right)^{12} \).

Example 3.2.9.

\[
[x^2(x+1)^3] [x^6(x+1)^{11}] = [x^2x^6][(x+1)^3(x+1)^{11}]
\]
\[
= x^{2+6}(x+1)^{3+11}
\]
\[
= x^8(x+1)^{14}.
\]

Example 3.2.10.

\[
(3.2)^2 \cdot (3.2)^3 \cdot (3.2)^6 \cdot (3.2) = (3.2)^{2+3} \cdot (3.2)^6 \cdot (3.2) \quad \text{Order of Operations}
\]
\[
= (3.2)^{2+3+6} \cdot (3.2)^1 \quad 3.2 = (3.2)^1
\]
\[
= (3.2)^{2+3+6+1}
\]
\[
= (3.2)^{12}.
\]

This computation illustrates the fact that we could have just added all of the exponents in the beginning:

\[
(3.2)^2 \cdot (3.2)^3 \cdot (3.2)^6 \cdot (3.2) = (3.2)^{2+3+6+1} = (3.2)^{12},
\]
as long as we remember that \( (3.2) = (3.2)^1 \).

Example 3.2.11. Simplify \( \frac{3^7}{3^9} \).

**Solution:** In order to simplify a fraction, we need to find factors that the numerator and denominator share. The numerator has a factor of \( 3^7 \), and so does the denominator since

\[
3^9 = 3^{2+7} = 3^2 \cdot 3^7.
\]

To see this, we have applied the Multiplication of Like Bases Theorem, Theorem 3.2.2, “backwards.” Now we may compute.
Example 3.2.12. Simplify $\frac{5^6}{5^2}$.

Solution: Notice that $5^6 = 5^25^4$. Thus,

$$\frac{5^6}{5^2} = \frac{5^25^4}{5^2} = 5^4 = 625.$$  

Notice that the final exponent, 4, is the difference between the original exponents: $6 - 2 = 4$. 

Example 3.2.13.

$$\frac{x^8}{x^{12}} = \frac{x^8}{x^4x^8} = \frac{1}{x^4}.$$  

Notice that the final exponent in the denominator, 4, is the difference between the original exponents: $12 - 8 = 4$. 

In both of the preceding examples, the final exponent was the difference between the original exponents. This makes sense: to simplify the fraction, we must cancel common factors. If the exponent in the numerator is greater than that in the denominator, then the entire denominator is cancelled, and what is left in the numerator is however many factors the denominator did not cancel. This idea is recorded in the following theorem.

**Theorem 11** (Division of Like Bases). Let $a$ be a real number, and let $m$ and $n$ be natural numbers. Then

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ 1 & \text{if } m = n \\ \frac{1}{a^{n-m}} & \text{if } n > m \end{cases}$$
Example 3.2.14.
\[
\frac{12^8}{12^5} = 12^{8-5} = 12^3.
\]
\hfill \square

Example 3.2.15.
\[
\frac{x^{14}}{x^{14}} = 1.
\]
\hfill \square

Example 3.2.16.
\[
\frac{(6x)^3}{(6x)^7} = \frac{1}{(6x)^{7-3}} = \frac{1}{(6x)^4} = \frac{1}{6^4x^4}.
\]
\hfill \square

Example 3.2.17.
\[
\frac{8x^5y^2}{4x^3y^9} = \frac{2x^{5-3}}{y^{9-2}} = \frac{2x^2}{y^7}.
\]
\hfill \square

Notice that in the last example, we had three different factors to worry about: a constant factor, a factor involving \(x\), and a factor involving \(y\). However, we saw in Chapter 2 that we can deal with each factor separately, and so that is what we did.

We now change gears slightly to consider another application of the Multiplication of Like Bases Theorem.

Example 3.2.18. What is \((4^5)^3\)\

**Solution:** To evaluate this expression, we need to expand the third power:

\[
(4^5)^3 = (4^5)(4^5)(4^5)
\]
\[
= [4^{5+5}](4^5)
\]
\[
= 4^{5+5+5}
\]
\[
= 4^{5\cdot3}
\]
\[
= 4^{15}.
\]

Thus, we have that \((4^5)^3 = 4^{5\cdot3} = 4^{15}\). Notice why the exponent was \(5 \cdot 3\): each of the three factors of \(4^5\) had a \(5\) in the exponent, so the three fives were added together. Usually, instead of adding three fives, we multiply \(5\) by \(3\).
\hfill \square
Example 3.2.19.

\[(5^7)^2 = (5^7)(5^7)\]

\[= 5^{7+7}\]

\[= 5^{7\cdot2}\]

\[= 5^{14}\].

\[\text{□}\]

Example 3.2.20.

\[(x^4)^5 = (x^4)(x^4)(x^4)(x^4)(x^4)\]

\[= x^{4+4+4+4+4}\]

\[= x^{4\cdot5}\]

\[= x^{20}\]

Again, we end up with 5 fours in the exponent by using the Multiplication Of Like Bases Theorem.

\[\text{□}\]

Example 3.2.21.

\[(a^m)^n = \underbrace{(a^m)(a^m)(a^m)\cdots(a^m)(a^m)}_{n \text{ factors of } a^m}\]

\[= \underbrace{a^m + a^m + \cdots + a^m + a^m}_{n \text{ terms}}\]

\[= a^{mn}\]

\[\text{□}\]

The preceding examples lead to the following theorem.

**Theorem 12** (Raising a Power (of a) to a Power). Let \(a\) be a real number and \(m\) and \(n\) natural numbers. Then

\[(a^m)^n = a^{mn}.\]

Example 3.2.22. \((72^5)^8 = 72^{5\cdot8} = 72^{40}\)

\[\text{□}\]

Example 3.2.23. \([3x^2]^7 = (3x^2)^{4\cdot7} = (3x^2)^{28}\).
So far, most of our bases have had only a single factor. What happens when the base has more than one factor?

Example 3.2.24.

\[(3 \cdot 4)^2 = (3 \cdot 4)(3 \cdot 4) = (3 \cdot 3)(4 \cdot 4) = 3^2 \cdot 4^2.\]

Example 3.2.25.

\[(3x)^3 = (3x)(3x)(3x) = (3 \cdot 3 \cdot 3)(x \cdot x \cdot x) \text{ (Multiplication is associative and commutative)} = 3^3x^3.\]

Example 3.2.26.

\[(xy)^4 = (xy)(xy)(xy)(xy) = (x \cdot x \cdot x \cdot x)(y \cdot y \cdot y \cdot y) = x^4y^4.\]

Example 3.2.27.

\[(ab)^n = (ab)(ab)(ab)\cdots(ab)(ab)\]

\[\phantom{\text{Example 3.2.27.}}\]

\[\text{ } = (a \cdot a \cdot a \cdot a \cdot a)(b \cdot b \cdot b \cdot b \cdot b)\]

\[\phantom{\text{Example 3.2.27.}}\]

\[\text{ } = a^n b^n.\]

Each factor appears as many times as the exponent indicates.

**Theorem 13** (Power of a Product). Let \(a\) and \(b\) be real numbers and \(n\) be a natural number. Then

\[(ab)^n = a^n b^n.\]

Example 3.2.28. \((4y)^5 = 4^5y^5.\)
Example 3.2.29.

\[-5x^2)^3 = (-5)^3(x^2)^3\]
\[= -125x^{2\cdot3}\]
\[= -125x^6\]

The last example illustrates that sometimes you will need to use more than one property to simplify an expression. Just keep the properties and the order of operations in mind as you do!

Example 3.2.30.

\[(4x^3)^5(3x^2)^6 = [4^5(x^3)^5][3^6(x^2)^6]\]
\[= [1024x^{3\cdot5}][729x^{2\cdot6}]\]
\[= (1024x^{15})(729x^{12})\]
\[= 746496x^{15\cdot12}\]
\[= 746496x^{15+12}\]
\[= 746496x^{27}\]

Example 3.2.31.

\[(x^3)^4(5x)^7 = x^{3\cdot4}(5^7x^7)\]
\[= 5^7x^{12\cdot7}\]
\[= 5^7x^{12+7}\]
\[= 5^7x^{19}\]

Since multiplication and division are very closely related, it seems reasonable that a similar rule should hold for powers of quotients as does for powers of products.
Example 3.2.32.

\[
\left(\frac{4}{7}\right)^3 = \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} = \frac{4 \cdot 4 \cdot 4}{7 \cdot 7 \cdot 7} = \frac{4^3}{7^3}.
\]

Example 3.2.33.

\[
\left(\frac{-2}{9}\right)^6 = \frac{-2}{9} \cdot \frac{-2}{9} \cdot \frac{-2}{9} \cdot \frac{-2}{9} \cdot \frac{-2}{9} \cdot \frac{-2}{9} = \frac{(-2)(-2)(-2)(-2)(-2)(-2)}{9 \cdot 9 \cdot 9 \cdot 9 \cdot 9} = \frac{(-2)^6}{9^6}.
\]

Example 3.2.34.

\[
\left(\frac{2}{3x}\right)^4 = \frac{2}{3x} \cdot \frac{2}{3x} \cdot \frac{2}{3x} \cdot \frac{2}{3x} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{(3x)(3x)(3x)(3x)} = \frac{2^4}{(3x)^4} = \frac{16}{3^4x^4} = \frac{16}{81x^4}.
\]

Example 3.2.35.

\[
\left(\frac{x}{y}\right)^5 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y} = \frac{x^5}{y^5}.
\]
Example 3.2.36.

\[
\left( \frac{a}{b} \right)^n = \left( \frac{a}{b} \right) \left( \frac{a}{b} \right) \cdots \left( \frac{a}{b} \right) \left( \frac{a}{b} \right) \\
= \left( \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} \right) \\
= \left( \frac{a}{b} \right)^n \\
= \frac{a^n}{b^n}.
\]

These examples illustrate the following theorem.

**Theorem 14** (Power of a Quotient). Let \( \frac{a}{b} \) be a fraction or rational expression and let \( n \) be a natural number. Then

\[
\left( \frac{a}{b} \right)^n = \frac{a^n}{b^n}.
\]

Example 3.2.37. \( \left( \frac{3}{8} \right)^5 = \frac{3^5}{8^5} \).

Example 3.2.38. \( \left( \frac{x}{x+1} \right)^8 = \frac{x^8}{(x+1)^8} \). We are making the implicit assumption that \( x \neq -1 \).

We record here for easy reference the rules we have developed thus far.

**Theorem 15.** Let \( a \) and \( b \) be real numbers and \( m \) and \( n \) natural numbers. Then

1. \( a^m a^n = a^{m+n} \) (Multiplication of like bases)
2. \( \frac{a^m}{a^n} = \begin{cases} 
    a^{m-n} & \text{if } m > n \\
    1 & \text{if } m = n \\
    a^{n-m} & \text{if } n > m 
\end{cases} \) (Division of like bases)
3. \( (a^m)^n = a^{mn} \) (Raising a power (of a) to a power)
4. \( (ab)^n = a^n b^n \) (Power of a Product)
5. \( \left( \frac{a}{b} \right)^n = \frac{a^n}{b^n} \) if \( b \neq 0 \) (Power of a Quotient)
You should practice with these rules until they become comfortable. While you may want to memorize them, it is far more important to understand them. That way, if you forget the rule, you can figure it out again for yourself just by considering the definition of exponents. Note that we will dramatically simplify the Division of Like Bases theorem in the next section by introducing negative exponents.

As a final example, let’s consider how we can use these rules on rational expressions.

**Example 3.2.39.**

\[
\frac{x^2(x + 3)^3}{(x + 1)(3x - 2)^2} \cdot \frac{x^4(3x - 2)^3}{(x + 3)(x + 1)^5} = \frac{x^2(x + 3)^3x^4(3x - 2)^3}{(x + 1)(3x - 2)^2(x + 3)(x + 1)^5}
\]

\[
= \frac{x^{2+4}(x + 3)^{3-1}(3x - 2)^{3-2}}{(x + 1)^{1+5}}, \quad x \neq \frac{2}{3}, x \neq -3
\]

\[
= \frac{x^6(x + 3)^2(3x - 2)}{(x + 1)^6}, \quad x \neq \frac{2}{3}, x \neq -3.
\]

\[\square\]

Notice that we used the Multiplication of Like Bases Theorem and the Division of Like Bases Theorem.
Section 3.2 Exercises

Simplify or expand as indicated.

1. \(4^8 \cdot 4^7\)
2. \(2^3 \cdot 2^9\)
3. \(\left(\frac{3}{8}\right)^2 \cdot \left(\frac{3}{8}\right)^7\)
4. \(1^{523985}\)
5. \(\frac{5^6}{5^3}\)
6. \(\frac{4^{12}}{4^{17}}\)
7. \((3^4)^7\)
8. \([-2^5]^4\)
9. \((2x)^4\)
10. \((5t)^7\)
11. \(\left(\frac{x}{3}\right)^6\)
12. \((x^2y)^4\)
13. \(\frac{x^4}{x^{13}}\)
14. \(\left(\frac{(x^3 + 3x + 1)^4}{(x^3 + 3x + 1)^9}\right)^5\)
15. \(24^3 \cdot 24^5 \cdot \frac{1}{24^8}\)
16. \(\frac{4^4}{24}\)
17. \(\frac{3x^2(x+2)^3}{9x^3(x+2)(x+1)^4(x+1)} \div \left(\frac{3(x+2)}{x+2}\right)^3\)
18. \(\frac{x^2}{x+2} - \frac{4x}{x+8}\)
19. \([(x^2y^3)^4]^6\)
20. \(x + \frac{7x + 2}{x^2}\)
21. \(\left(-\frac{5x}{8}\right)^5\)
22. \(\frac{21x^4}{54x^8}\)
23. \((abcde)^5\)
24. \(\frac{25(2t^2 + 3t + 1)^4}{12(t^3 - 8)^2} \div \left(\frac{30(t^3 - 8)}{(2t^2 + 3t + 1)^3}\right)\)
25. \((7x \cdot 4y)^5\)
26. \(\frac{18(x+3)^4}{12x^6(x+3)^4}\)
27. \(\frac{x^2 + 1}{3x^3} - \frac{5x + 6}{15x^2}\)
28. \(\left(\frac{t(t + 1)^2}{t + 2}\right)^{14}\)
29. \(5x^8 \cdot x^4 \cdot x^3\)
30. \(-x^3 + \frac{x^4}{x + 7}\)
31. \(\frac{t^2 + 1}{(t + 1)^2} \div \frac{4}{(t + 1)^3(t^2 + 1)^3}\)
32. \((3x^4)^3(2x^2)^5\)
33. \(\frac{(3x)^6y^7z^3(x + 2)^2}{3x^6y^3z^5(x + 2)^4}\)
34. \(\frac{x^3(x^2 + 4)^3}{x(x^2 - 2)^4} \div \frac{x^2(x^2 + 4)^6}{x^2 - 2}\)
35. \(\frac{x}{2x + 1} + \frac{5x}{x - 3}\)
36. \(\frac{(x + 2)^3}{(x + 1)(x + 3)^2} \div \frac{(x + 2)^4(x + 1)}{(x + 3)^2}\)
37. \(\left(\frac{x^2y^4}{xy^6}\right)^4\)
38. \(\frac{4x^3(2x^2 - 1)^4}{x + 2} \div \frac{x + 2}{2x(3x^2 - 1)^8}\)
39. Explain why $3^2 \cdot 3^4 \neq 9^6$.

### 3.3 Integer Exponents

In the preceding section, we saw that

$$\frac{a^m}{a^n} = \begin{cases} 
  a^{n-n} & \text{if } m > n \\
  1 & \text{if } m = n \\
  \frac{1}{a^{n-m}} & \text{if } n > m
\end{cases}$$

It would be much more convenient if we did not have three cases to deal with every time we wanted to apply our theorem. Accordingly, we now make some definitions.

**Definition 3.3.1.** Let $a$ be a nonzero real number, and let $n$ be a natural number.

1. $a^0 = 1$.
2. $a^{-n} = \frac{1}{a^n}$.
3. $\frac{1}{a^{-n}} = a^n$.

At first glance, these definitions may not seem to help much, but we will see that they are very convenient.

**Example 3.3.1.** Why should $a^0 = 1$? Consider the following pattern:

$$2^4 = 16, \quad 2^3 = 8, \quad 2^2 = 4, \quad 2^1 = 2, \quad 2^0 = 1.$$ 

As the exponents go down, the value is cut in half each time so that when we get to an exponent of 0, the value should be 1. In addition, notice that in the theorem if $m = n$, then we have $\frac{a^m}{a^n} = 1$, so if we use the Division of Like Bases idea and subtract the exponents, we have $a^{m-n} = 1$, or $a^0 = 1$.

**Example 3.3.2.** $12^0 = 1$.

Notice that

$$12^3 = 1728, \quad 12^2 = 144, \quad 12^1 = 12, \quad 12^0 = 1.$$ 

Each value is one-twelfth of the value preceding it.

**Example 3.3.3.** $4^{-3} = \frac{1}{4^3}$. 

□
Example 3.3.4. \( \frac{1}{3^{-5}} = 3^5 \).

Example 3.3.5.

\[
\left( \frac{2}{5} \right)^{-3} = \left( \frac{1}{\frac{2}{5}} \right)^3
\]

\[
= \left( \frac{5}{2} \right)^3
\]

\[
= \frac{5^3}{2^3}
\]

\[
= \frac{125}{8}.
\]

Notice that

\[
\left( \frac{2}{5} \right)^{-3} = \left( \frac{5}{2} \right)^3.
\]

That is, the negative exponent indicates a reciprocal.

Example 3.3.6.

\[
\frac{x^4}{x^6} = \frac{1}{x^{6-4}}
\]

\[
= \frac{1}{x^2}
\]

\[
= x^{-2}.
\]

Example 3.3.7.

\[
\frac{t^5}{t^{-2}} = \frac{t^5}{1} \cdot \frac{1}{t^{-2}}
\]

\[
= t^5 t^2
\]

\[
= t^7.
\]

Here, we separated the numerator and denominator by using the Fraction Multiplication Theorem “backwards”. This allowed us to deal with the \( \frac{1}{t^2} \) according to the definition. Also, the final exponent 7 is equal to \( 5 - (-2) \), the difference in the original exponents.
Example 3.3.8.

\[
\frac{5x^2y^6}{x^{-3}y^6} = \frac{1}{x^{-3}} \frac{5x^2y^6}{y^6} = x^3 \cdot 5x^2 = 5x^{3+2} = 5x^5.
\]

We separated out the factor of \( x^{-3} \) in the denominator so that we could handle it separately. Also notice that the final exponent 5 on \( x \) is \( 2 - (-3) \), once more the difference between the exponents in the numerator and denominator.

Example 3.3.9.

\[
\frac{x^{-3}}{x^4} = \frac{x^{-3}}{1} \cdot \frac{1}{x^4} = \frac{1}{x^3} \cdot \frac{1}{x^4} = \frac{1}{x^3 \cdot x^4} = \frac{1}{x^7}.
\]

Example 3.3.10. \( \frac{3^7}{3^7} = 1 \). Notice that \( 3^0 = 1 \) as well, so the final exponent is once more the difference between the exponents in the numerator and denominator: \( 7 - 7 = 0 \).

These examples indicate how the Division of Like Bases Theorem should be modified.

**Theorem 16.** Let \( a \) be a nonzero real number and let \( m \) and \( n \) be integers. Then

\[
\frac{a^m}{a^n} = a^{m-n}.
\]

Example 3.3.11.

\[
\frac{4^7}{4^6} = 4^{7-6} = 4^1 = 4.
\]
Example 3.3.12.

\[
\frac{2^7}{2^9} = 2^{7-9} = 2^{-2}.
\]

Example 3.3.13.

\[
\frac{x^4y^2z}{xy^5z^3} = \frac{x^4y^2z^1}{x^1y^5z^3} = x^{4-1}y^{2-5}z^{1-3} = x^3y^{-3}z^{-2}.
\]

Example 3.3.14. We know from our discussion of fraction simplification that

\[
\frac{15^{11}}{15^{11}} = 1.
\]

We also have

\[
\frac{15^{11}}{15^{11}} = 15^{11-11} = 15^0 = 1,
\]

so we see that this theorem is consistent with what we have done before.

The exponent laws we discovered in the last section hold for all integer exponents, not just natural number exponents.

**Theorem 17.** Let \(a\) and \(b\) be real numbers, and let \(m\) and \(n\) be integers. Then

1. \(a^m a^n = a^{m+n}\) (Multiplication of like bases)

2. \(\frac{a^m}{a^n} = a^{m-n}\) (Division of like bases)

3. \((a^m)^n = a^{mn}\) (Raising a power (of \(a\)) to a power)

4. \((ab)^n = a^n b^n\) (Power of a Product)
5. \((\frac{a}{b})^n = \frac{a^n}{b^n}\) (Power of a Quotient), provided that all appropriate quantities are defined.

Example 3.3.15.

\[
(-8)^7(-8)^{-4} = (-8)^{7+(-4)} \\
= (-8)^3 \\
= -512.
\]

□

Example 3.3.16.

\[
(-4)^5(4)^{-6} = (-1)^5 \cdot 4^5 \cdot 4^{-6} \\
= -1 \cdot 4^{5+(-6)} \\
= -4^{-1} \\
= -\frac{1}{4}.
\]

□

Example 3.3.17.

\[
\left[\left(\frac{-3}{11}\right)^3\right]^{-4} = \left(\frac{-3}{11}\right)^{3(-4)} \\
= \left(\frac{-3}{11}\right)^{-12} \\
= \left(\frac{11}{-3}\right)^{12} \\
= 11^{12} \cdot (-3)^{12} \\
= \frac{11^{12}}{3^{12}}.
\]

□
Example 3.3.18.

\[
\left( \frac{3x^3}{5xy} \right)^{-2} = \left( \frac{3x^{3-1}}{5y} \right)^{-2} \\
= \left( \frac{3x^2}{5y} \right)^{-2} \\
= \left( \frac{5y}{3x^2} \right)^2 \\
= \frac{(5y)^2}{(3x^2)^2} \\
= \frac{5^2y^2}{3^2(x^2)^2} \\
= \frac{25y^2}{9x^4} \\
= \frac{25y^2}{9x^4}.
\]

Whew! That one took a lot of different properties.

\[\square\]

Example 3.3.19. Notice that

\[
x \cdot x^{-1} = x^1 \cdot x^{-1} \\
= x^{1-1} \\
= x^0 \\
= 1,
\]

so \(x^{-1}\) is another way to write \(\frac{1}{x}\), the multiplicative inverse of \(x\).

\[\square\]

Section 3.3 Exercises

Express each quantity without negative exponents.

1. \(3^{-3}\)  
2. \(5^{-2}\)  
3. \(\frac{1}{3^{-4}}\)  
4. \(x^{-3}\)  
5. \(\frac{4}{x^{-2}}\)
6. \( \frac{(x + 1)^{-4}}{(x^2 - 2)^{-2}} \) \\
7. \( 14^{-11} \) \\
8. \( \frac{1}{(x + 3)^{-6}} \) \\
9. \( \frac{4^{-3}}{5^{-1}} \) \\
10. \( x^{-1} \) \\
11. \( t^{-4} \) \\
12. \( \frac{6^2}{x^{-3}} \)

Compute as indicated and simplify as much as possible.

13. \( \frac{2^{-4}}{2^3} \) \\
14. \( (x^5 + 7x^2 - 38x + 12)^0 \) \\
15. \( 4^{-3} \cdot 4^{-12} \) \\
16. \( \frac{3x^{-4}}{6x^6} \) \\
17. \( x^2x^{-4} \) \\
18. \( \frac{x^4y^{-3}}{x^{-1}y^2} \) \\
19. \( \left( \frac{x^3y^{-2}}{x^{-1}y^{-3}} \right)^5 \) \\
20. \( (4x^2)^{-3} \) \\
21. \( \left( \frac{x^4}{y^3} \right)^{-2} \) \\
22. \( \frac{x^{-3}(x + 2)^3}{x^2(3x + 1)^{-4}} \cdot \frac{x^2(x + 2)(3x + 1)^{-2}}{x^{-5}(x + 2)^{-4}} \) \\
23. \( \frac{x^{-2}y^3}{x^4y^2} \) \\
24. \( \frac{(3x^2 + 2)^4}{(x + 1)^{-2}} \div \frac{(3x^2 + 2)^{-1}}{(x + 1)^2} \)

### 3.4 Rational Exponents

In this section, we extend our definition of exponents to include rational exponents.

**Definition 3.4.1.** Let \( a \) be a real number and let \( n \) be an integer. Then \( a^{\frac{1}{n}} \) is defined by

\[
a^{\frac{1}{n}} = b \text{ if and only if } a = b^n
\]

for the largest such \( b \) that exists, if there is one. If there is such a \( b \) it is called the **principal \( n \)th root** of \( a \).

**Example 3.4.1.** \( 9^{\frac{1}{2}} = 3 \) since \( 9 = 3^2 \). Also, \((-3)^2 = 9\), so which is the principal root? The definition above says to take the largest number that works. Since only 3 and -3 satisfy \( x^2 = 9\), the principal root is 3.

Notice that

\[
9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}} = 3 \cdot 3 = 9 \text{ and } 9^{\frac{1}{2} + \frac{1}{2}} = 9^1 = 9.
\]

Also notice that

\[
(9^{\frac{1}{2}})^2 = 3^2 = 9 \text{ and } 9^{\frac{1}{2} \cdot 2} = 9^1 = 9.
\]

**Example 3.4.2.** \( 121^{\frac{1}{2}} = 11 \) since \( 121 = 11^2 \). Again, \((-11)^2 = 121\), but 11 is larger than -11, so it is our principal root.
In both of the preceding examples, the exponent was $\frac{1}{2}$. We call $a^{1/2}$ the (principal) square root of $a$.

*Example 3.4.3.* $8^{1/3} = 2$ since $8 = 2^3$. Notice that 2 is the only number such that satisfies $x^3 = 8$, so there is no question about what the principal root is.

*Example 3.4.4.* $216^{1/3} = 6$ since $216 = 6^3$.

In the two preceding examples, the exponent was $\frac{1}{3}$. We call $a^{1/3}$ the cube root of $a$.

*Example 3.4.5.* Suppose that $x^{\frac{1}{3}} = y$. Then $x = y^3$. Substituting, we see that

$$x = \left(\frac{1}{3}ight)^3.$$

In the previous sections, we saw that to raise a power of a base to another power, we multiplied exponents. This still works here, too!

*Example 3.4.6.* $(16x^4)^{\frac{1}{4}} = 2x$ if $x$ is positive since

$$(2x)^4 = 2^4x^4 = 16x^4,$$

so

$$(16x^4)^{\frac{1}{4}} = \left[(2x)^4\right]^{\frac{1}{4}}.$$

We had to assume that $x$ was positive in the above example in order to make sure that we found the principal root. Throughout the rest of this section, we will assume that variables are positive in order to simplify the exposition. If the variables are allowed to be negative, the formulas are slightly more complicated. It is, therefore, extremely important to keep in mind that all variables are assumed to be positive.

*Example 3.4.7.* $(-4)^{\frac{1}{2}}$ does not exist since there is no real number $b$ such that $b^2 = -4$. (Recall that the square of any real number is at least 0.) Therefore, the symbol $(-4)^{\frac{1}{2}}$ is undefined.

*Example 3.4.8.* What is the 5th root of 1024?

**Solution:** Since $4^5 = 1024$, the fifth root of 1024 is 4; that is, $1024^{\frac{1}{5}} = 4$. 
Example 3.4.9.

\[ 8^{\frac{1}{3}} = \frac{1}{2} \]
since

\[ \left( \frac{1}{2} \right)^{-3} = \left( \frac{1}{2} \right)^3 = 2^3 = 8. \]

\[ \square \]

It is not hard to check whether a given root is correct; for example, above, we just had to compute \( (\frac{1}{2})^{-3} \) and see whether we ended up with 8. But where does the number to check come from? It comes from experience, educated guessing, and trial and error. You will need to practice to become good at it, and we will also learn some rules to help us quickly identify candidates.

Example 3.4.10.

\[ (-8)^{\frac{1}{3}} = -2 \text{ since } (-2)^3 = 8. \]

\[ \square \]

Example 3.4.11.

\[ (x^6)^{\frac{1}{3}} = x^2 \text{ since } (x^2)^3 = x^6 = x^6. \]

\[ \square \]

Example 3.4.12.

\[ \left( \frac{x^4 y^8}{16} \right)^{\frac{1}{2}} = \frac{x^2 y^4}{4} \]
since

\[ \left( \frac{x^2 y^4}{4} \right)^{\frac{1}{2}} = \frac{(x^2)^{\frac{1}{2}}(y^4)^{\frac{1}{2}}}{4^{\frac{1}{2}}} \]

\[ = \frac{x^{2 \cdot \frac{1}{2}} y^{4 \cdot \frac{1}{2}}}{16} \]

\[ = \frac{x^2 y^2}{16}. \]

\[ \square \]

We now define what is meant by a rational exponent.

**Definition 3.4.2.** Let \( a \) be a real number, and let \( \frac{m}{n} \) be a fraction. Then

\[ a^{\frac{m}{n}} = \left( a^{\frac{1}{n}} \right)^m. \]

We will see that this definition allows us to keep all of the exponent properties we have already discovered.
Example 3.4.13.

\[ 8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^{2} = 2^{2} = 4. \]

Here, we used Example 3.4.3 for the second step.

Example 3.4.14.

\[ x^{\frac{2}{3}}x^{\frac{1}{3}} = \left(x^{\frac{1}{3}}\right)^{2} \left(x^{\frac{1}{3}}\right)^{1} \]
\[ = \left(x^{\frac{1}{3}}\right)^{2+1} \text{ Multiplication of Like Bases} \]
\[ = \left(x^{\frac{1}{3}}\right)^{3} \]
\[ = \left(x^{\frac{4}{3}}\right) \text{ Definition} \]
\[ = x^{1} \]
\[ = x. \]

Example 3.4.15. In the previous example, the exponents had a common denominator. When we applied the definition of a rational exponent, that ended up giving us like bases of \(x^{\frac{1}{3}}\). Here, however, the exponents do not have a common denominator, so we must first find one.

\[ 16^{\frac{3}{8}} \cdot 16^{\frac{1}{3}} = 16^{\frac{2}{3} + \frac{1}{3}} \]
\[ = \left(16^{\frac{1}{3}}\right)^{3} \left(16^{\frac{1}{3}}\right)^{4} \]
\[ = \left(16^{\frac{1}{3}}\right)^{3+4} \text{ Multiplication of Like Bases} \]
\[ = \left(16^{\frac{1}{3}}\right)^{7} \]
\[ = 16^{\frac{7}{3}}. \]

Example 3.4.16.

\[ 5^{\frac{3}{5}} \cdot 5^{\frac{4}{5}} = 5^{\frac{3+4}{5}} \]
\[ = 5^{\frac{7}{5}} \cdot 5^{\frac{4}{5}} \]
\[ = \left(5^{\frac{1}{5}}\right)^{7} \left(5^{\frac{1}{5}}\right)^{20} \]
\[ = \left(5^{\frac{1}{5}}\right)^{27+20} \text{ Multiplication of Like Bases} \]
\[ = \left(5^{\frac{1}{5}}\right)^{47} \]
\[ = 5^{\frac{47}{20}} \text{ Definition.} \]
Example 3.4.17.

\[
(27^{\frac{3}{4}})^4 = \left( (27^{\frac{3}{4}})^{2} \right)^4 \\
= (27^{\frac{3}{4}})^2 \cdot 4 \quad \text{Raising a Power to a Power} \\
= (27^{\frac{3}{4}})^8 \\
= (3)^8 \\
= 6561.
\]

Example 3.4.18.

\[
\left( t^{\frac{3}{4}} \right)^{\frac{1}{2}} = t^{\frac{3}{8}}
\]

since

\[
\left( t^{\frac{3}{8}} \right)^5 = \left[ (t^{\frac{3}{8}})^3 \right]^5 \\
= \left[ t^{\frac{3}{8}} \right]^{3 \cdot 5} \\
= \left[ t^{\frac{3}{8}} \right]^{15} \\
= t^{\frac{15}{8}} \\
= t^{\frac{3}{16}}.
\]

Notice that

\[
\frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}
\]

Example 3.4.19.

\[
(25x^4)^{\frac{2}{3}} = \left[ (25x^4)^{\frac{1}{3}} \right]^3 \\
= [5x^2]^3 \quad \text{(See below)} \\
= 5^3(x^2)^3 \quad \text{Power of a Product} \\
= 125x^6.
\]

In the second step, we get

\[
(25x^4)^{\frac{1}{2}} = 5x^2
\]
because

\[(5x^2)^2 = 5^2(x^2)^2\]
\[= 25x^{2\cdot2}\]
\[= 25x^4.\]

\[\square\]

In the next example, we will compare two different methods for computing a quantity raised to a rational power.

**Example 3.4.20.** What is \(4^{\frac{3}{2}}\)?

**Solution:**

\[4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8.\]

On the other hand,

\[\left(4^3\right)^{\frac{1}{2}} = 64^{\frac{1}{2}} = 8,\]

which is the same result we found the first way.

\[\square\]

This property is true in general.

**Theorem 18.** Let \(a\) be a real number, and let \(m\) and \(n\) be integers, with \(n \neq 0\), such that \(a^{\frac{1}{n}}\) is defined. Then

\[a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}.\]

The preceding examples illustrate the fact that the properties of exponents we learned before are preserved even when the exponents are rational. The following theorem summarizes all of those properties.

**Theorem 19.** Let \(a\) and \(b\) be real numbers, and let \(m\) and \(n\) be rational numbers. Then

1. \(a^m a^n = a^{m+n}\) (Multiplication of like bases)
2. \(\frac{a^m}{a^n} = a^{m-n}\) (Division of like bases)
3. \((a^m)^n = a^{mn}\) (Raising a power (of a) to a power)
4. \((ab)^n = a^n b^n\) (Power of a Product)
5. \(\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\) (Power of a Quotient)
6. \(a^{-m} = \frac{1}{a^m}\)
7. \(\frac{1}{a^{-m}} = a^m\)
8. \(a^0 = 1\) if \(a \neq 0,\)

provided that all of the necessary quantities are defined.

**Section 3.4 Exercises**

Evaluate each expression.
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$4^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>2.</td>
<td>$8^{\frac{1}{4}}$</td>
</tr>
<tr>
<td>3.</td>
<td>$27^{\frac{1}{6}}$</td>
</tr>
<tr>
<td>4.</td>
<td>$\frac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}}$</td>
</tr>
<tr>
<td>5.</td>
<td>$64^{\frac{5}{6}}$</td>
</tr>
<tr>
<td>6.</td>
<td>$\left(\frac{9}{16}\right)^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>7.</td>
<td>$(-32)^{\frac{2}{5}}$</td>
</tr>
<tr>
<td>8.</td>
<td>$(-8)^{\frac{1}{3}}$</td>
</tr>
<tr>
<td>9.</td>
<td>$\left(\frac{8}{27}\right)^{-\frac{1}{3}}$</td>
</tr>
<tr>
<td>10.</td>
<td>$125^{\frac{4}{5}}$</td>
</tr>
<tr>
<td>11.</td>
<td>$\left(\frac{16}{81}\right)^{\frac{3}{4}}$</td>
</tr>
<tr>
<td>12.</td>
<td>$121^{\frac{1}{2}}$</td>
</tr>
</tbody>
</table>
Compute as indicated and simplify as much as possible. Assume that all variables are positive.

13. \( (x^4)^{1/2} \)

14. \( (16x^{1/2})^{3/2} \)

15. \( (49x^6)^{2/3} \)

16. \( t^{2/3}t^{-2/3} \)

17. \( y^{5/7}y^{1/3} \)

18. \( x^{3/4}x^{3/4} \)

19. \( \frac{x^{3/4}}{x^{10}} \)

20. \( \frac{x^{-3/5}}{x^3} \)

21. \( (x^2y)^{3/2} \)

22. \( (x^{1/5}y^{-1/3})^{15} \)

23. \( \left( \frac{x^{-3/4}y^{3/5}}{x^{1/2}y^{-1/4}} \right)^{-4} \)

24. \( (25x^4y^6)^{1/3} \)

### 3.5 Radical Expressions

For completeness, we describe another way of denoting rational exponents. We will assume throughout this section (as we did in the previous section) that variables are restricted to nonnegative values. Note that if we do not restrict to nonnegative variables, then \( \sqrt{(-7)^2} = \sqrt{49} = 7 \neq -7 \).

**Definition 3.5.1.** Let \( a \) be a real number, and let \( m \) and \( n \) be integers, with \( n \neq 0 \). Then

\[
a^{\frac{m}{n}} = \sqrt[n]{a^m}.
\]

The symbol \( \sqrt[n]{\cdot} \) is called a radical. With this notation, the integer \( n \) appearing in the radical is called the index. The plural of index is indices. In the expression \( \sqrt[n]{a} \), \( a \) is called the radicand.

**Example 3.5.1.** \( x^{\frac{2}{3}} = \sqrt[3]{x^2} \).

**Example 3.5.2.** \( \sqrt[3]{8} = 8^{\frac{1}{3}} = 2 \).

**Example 3.5.3.** \( \sqrt[3]{64} = 64^{\frac{1}{3}} = 8 \). Usually, if the index is 2 we simply omit it. Thus,

\[
\sqrt{64} = \sqrt{64},
\]

which is the square root of 64.

**Example 3.5.4.** \( \sqrt{49} = 7 \) since \( 7^2 = 49 \).
Example 3.5.5.

\[ \sqrt[4]{\frac{x^8}{y^{12}}} = \left( \frac{x^8}{y^{12}} \right)^{\frac{1}{4}} \]

\[ = \frac{x^{8\cdot\frac{1}{4}}}{y^{12\cdot\frac{1}{4}}} \quad \text{Division of Like Bases} \]

\[ = \frac{x^2}{y^3}. \]

For the most part, we will be using rational exponents in this book rather than radicals. The primary exception is square roots. We will use \( \sqrt{x} \) throughout this book; other indices will appear much less frequently, if at all. The primary advantage to radicals is that their appearance is tidier than that of rational exponents.

The properties we know for rational exponents have the following analogues in radicals.

**Theorem 20.** Let \( a \) and \( b \) be real numbers, and let \( m \) and \( n \) be natural numbers. Then

1. \( \sqrt[n]{a^m} = (\sqrt[n]{a})^m \) if \( \sqrt[n]{a} \) is defined.

2. \( \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b} \) if \( \sqrt[n]{a} \) and \( \sqrt[n]{b} \) are both defined.

3. \( \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \) if \( \sqrt[n]{a} \) and \( \sqrt[n]{b} \) are both defined and \( b \neq 0 \).

4. \( \sqrt[n]{a^n} = a \) if \( a \) is nonnegative.

**Example 3.5.6.** \( \sqrt[3]{14^3} = 14 \).

**Example 3.5.7.** \( \sqrt{(41x)^2} = 41x \). (Recall: \( x \geq 0 \)).

**Example 3.5.8.** \( \sqrt{121} = 11 \) since \( 11^2 = 121 \). Also, \( -\sqrt{121} = -11 \). The square root symbol acts as a grouping symbol, too.

**Example 3.5.9.** \( \sqrt[3]{216} = 6 \) since \( 6^3 = 216 \).
Example 3.5.10. In order to find \( \sqrt[3]{3240} \), we first need to factor 3240. We have

\[ 3240 = 2^3 \cdot 3^4 \cdot 5. \]

Thus

\[
\sqrt[3]{3240} = \sqrt[3]{2^3 \cdot 3^4 \cdot 5} \\
= \sqrt[3]{2^3} \cdot \sqrt[3]{3^4} \cdot \sqrt[3]{5} \\
= 2 \cdot 3 \sqrt[3]{15} \\
= 6 \sqrt[3]{15}.
\]

Notice how we used the properties of the theorem to separate the factors into pieces we could handle.

□

Example 3.5.11.

\[ \sqrt{125} = \sqrt{5^2 \cdot 5} \\
= 5 \sqrt{5} \]

Here we first found the largest perfect square factor of 125 we could (namely, 25).

□

Example 3.5.12. \( \sqrt{x^3 y^2} = x^2 y \). (Recall: \( x, y \geq 0 \).)

□

Example 3.5.13.

\[ \sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} \\
= \sqrt[3]{x^3} \cdot \sqrt[3]{x} \\
= x \sqrt[3]{x}. \]

Notice that we needed to split the radicand into two pieces: the piece with a radicand of \( x^3 \) had an exponent that was divisible by the index, and the other had an exponent smaller than the index. This is because of our definitions:

\[
\sqrt[3]{x^3} = \left( x^3 \right)^{\frac{1}{3}} \\
= x^{3 \cdot \frac{1}{3}} \\
= x^1 \\
= x.
\]

Another way of looking at this is to observe that we found the largest perfect cube factor of \( x^4 \), which is \( x^3 \).
Example 3.5.14.

\[
\sqrt{x^7 y^3} = \sqrt{x^6 xy^2 y} = \sqrt{x^6 y^2 \sqrt{xy}} = x^3 y \sqrt{xy}.
\]

Again, the idea was to separate out as much of the radicand as possible while leaving the exponents as integers. We accomplished this by finding the largest perfect square factor of \(x^7 y^3\) we could.

Example 3.5.15.

\[
\sqrt[3]{27x^6(x + 1)^5} = \sqrt[3]{27 \sqrt[3]{x^6} \sqrt[3]{(x + 1)^5}} = 3x^2 \sqrt[3]{(x + 1)^3(x + 1)^2} = 3x^2(x + 1)^2.
\]

Example 3.5.16.

\[
\sqrt[6]{x^2 y^3} = \sqrt[6]{x^2} \sqrt[6]{y^3} = \sqrt[3]{x} \sqrt[2]{y}.
\]

In this example, we ended up with radicals having different indices. This is sort of in bad taste, as it makes it more difficult to see what we have, rather than less. It isn’t wrong to have different indices, but neither is it attractive, so we will avoid it when we simplify.

Section 3.5 Exercises

Convert each expression into a radical expression.

13. \(3^{\frac{1}{4}}\)  
14. \(15^{\frac{1}{3}}\)  
15. \(x^{\frac{7}{15}}\)  
16. \((2x)^{\frac{3}{11}}\)  
17. \((\frac{2}{x + 1})^{\frac{1}{4}}\)  
18. \((x^2)^{\frac{1}{4}}\)  
19. \((x^4)^{\frac{1}{5}}\)  
20. \((x^4)^{\frac{1}{6}}\)  
21. \((x^\frac{1}{3})^{\frac{7}{5}}\)  
22. \((x^4)^{\frac{1}{8}}\)  
23. \(x^{\frac{3}{5}}\)  
24. \(t^{-\frac{7}{3}}\)
Convert each radical expression into an expression with rational exponents.

25. \(\sqrt[4]{x^5}\)  
26. \(\sqrt[3]{x^3}\)  
27. \(\sqrt[6]{x^5}\)  
28. \(\sqrt[5]{2x^3}\)  
29. \(\sqrt[5]{x^5}\)  
30. \(\sqrt[7]{x^7}\)  
31. \(\sqrt{x}\)  
32. \(\sqrt{x^4y^3}\)  
33. \(\sqrt[4]{y^2}\)  
34. \(\sqrt[6]{16}\)  
35. \(\sqrt[11]{4x^{11}}\)  
36. \(\sqrt[7]{25(x+1)^7}\)

Identify the radicand and the index in each radical expression. Then simplify as much as possible.

37. \(\sqrt[6]{x^6}\)  
38. \(\sqrt[3]{64x^6}\)  
39. \(\sqrt{25x^4}\)  
40. \(\sqrt[2]{x^2}\)  
41. \(\sqrt[5]{5}\)  
42. \(\sqrt[3]{5}\)  
43. \(\sqrt{15x^3}\)  
44. \(\sqrt{75}\)  
45. \(\sqrt{x^{11}}\)  
46. \(\sqrt[5]{54x^8}\)  
47. \(\sqrt{y^9}\)  
48. \(\sqrt[3]{8}\)

Compute as indicated and simplify.

49. \(\sqrt[5]{48x^8y^5}\)  
50. \(\sqrt[4]{\frac{64}{(2x+1)^{15}}}\)  
51. \(\sqrt[6]{54t^6}\)  
52. \(\sqrt[8]{x^8}\)  
53. \(\sqrt[35]{x}\)  
54. \(\sqrt[11]{40t^11}\)  
55. \(\sqrt[4]{\frac{x^2+1}{16x^4}}\)  
56. \(\sqrt[3]{\frac{x^4(x+2)^7}{8x^2(x+2)^3}}\)  
57. \(\sqrt[5]{\frac{81}{x^5}}\)  
58. \(\sqrt[13]{x^6y^{13}(x^2+4)^3}\)  
59. \(\sqrt{3000}\)  
60. \(\sqrt[49]{250x^5}\)

### 3.6 Scientific Notation

When numbers are very large, it can be inconvenient to express them in the usual way. For example, it is hard to see precisely how big

\[246234510000000000000000\]

is. Scientific notation is a way of writing very large or very small numbers in a compact and easy to read form.

Scientific notation relies on our base-10 positional system. Recall that the number 1234.56 means 1 thousand, two hundreds, 3 tens, 4 ones, 5 tenths, and 6 hundredths. The value of each position is exactly 10 times the value of the position to its immediate right, and the value of the position directly to the left of the decimal point is one.
This makes it very easy to compute 10 (or 100 or 1000 or...) times a given number:

\[
(1234.56)(10) = 12345.6 \quad (= 1234.56 \times 10^1)
\]
\[
(1234.56)(100) = 123456 \quad (= 1234.56 \times 10^2)
\]
\[
(1234.56)(1000) = 1234560 \quad (= 1234.56 \times 10^3).
\]

The effect is just to shift the decimal point further to the right. Notice that the number of places the decimal point is shifted is equal to the exponent on the 10 shown to the far right.

Likewise, it is easy to compute \( \frac{1}{10} \) (or \( \frac{1}{100} \) or \( \frac{1}{1000} \) or ...) of a given number:

\[
(1234.56) \cdot \frac{1}{10} = 123.456 \quad (= 1234.56 \times 10^{-1})
\]
\[
(1234.56) \cdot \frac{1}{100} = 12.3456 \quad (= 1234.56 \times 10^{-2})
\]
\[
(1234.56) \cdot \frac{1}{1000} = 1.23456 \quad (= 1234.56 \times 10^{-3}).
\]

Again, the decimal point is just shifted, this time to the left. Also, the number of places it is shifted is just the size of the exponent on the 10. (Incidentally, it is called a decimal point because our system is base 10, corresponding to the Latin prefix *deci-*.)

Scientific notation takes advantage of these principles.

**Definition 3.6.1.** A number is said to be expressed in **scientific notation** if it is of the form

\[ x \times 10^n, \]

where \( 1 \leq x < 10 \) and \( n \) is an integer.

**Example 3.6.1.** The number

\[ 3.367 \times 10^4 \]

is in scientific notation, but

\[ 33.67 \times 10^3 \]

is not since 33 is greater than 10. Notice that these two numbers are equal:

\[ 3.367 \times 10^4 = 33670 \quad \text{and} \quad 33.67 \times 10^3 = 33670. \]

However, only the first one is expressed in scientific notation.

\[ \square \]

**Example 3.6.2.** The number \( 0.644 \times 10^3 \) is not in scientific notation since \( 0.644 < 1 \). However, \( 6.44 \times 10^2 \), which is equal to the original number, is in scientific notation.

\[ \square \]

**Example 3.6.3.** We have that \( 5.37 \times 10^{-5} = 0.0000537 \). To see this, simply shift the decimal point 5 places to the left.

\[ \square \]
Example 3.6.4. Express 2,346,000,000 in scientific notation.

Solution: If we shift the decimal point 9 places to the left, we will have 2.346, which is between 1 and 10, as needed. However, that isn’t the same number we started with, so we need to give instructions about how to recover the original number. Since we found 2.346 by shifting the decimal point 9 places left, we recover the original number by shifting 9 places to the right, which we can indicate by

\[2.346 \times 10^9.\]

Example 3.6.5. Express 0.00000078326 in scientific notation.

Solution: If we shift the decimal 7 places to the right, we have 7.8326, which is between 1 and 10. We can recover the original number by shifting back 7 places to the left, which we indicate by

\[7.8326 \times 10^{-7}.\]

Example 3.6.6. The rest mass of a proton, one of the constituent parts of the nucleus of an atom, is \(1.67 \times 10^{-27}\) kilograms. Very small! The rest mass of an electron is \(9.11 \times 10^{-31}\) kilograms, about \(\frac{1}{1800}\) that of the proton! On the other hand, the mass of the Earth is about \(5.98 \times 10^{24}\) kilograms.

None of these quantities is easily expressed in our standard notation, but they are easily expressed in scientific notation. [Source: Fundamentals of Physics, Third Edition, Halliday and Resnick, Wiley (1988).]

Example 3.6.7. The speed of light in a vacuum is about \(3.00 \times 10^8\) meters per second, or 300,000,000 meters per second.
Section 3.6 Exercises

Convert each number to scientific notation.

1. 341.2  
2. 0.013  
3. 58143  
4. 31,400,000,000  
5. 0.000003  
6. 0.00014  
7. 1.683  
8. 11.4  
9. 469,000,000,000,000  
10. 445  
11. 7,900,000  
12. 0.00000000000036  
13. $(4.5 \times 10^6)(3.8 \times 10^3)$

Express each number in standard form.

15. $4.6 \times 10^3$  
16. $5.9 \times 10^{-5}$  
17. $6.315 \times 10^{-2}$  
18. $5.83 \times 10^8$  
19. $6.4 \times 10^0$  
20. $3.857 \times 10^1$  
21. $8.23 \times 10^{-2}$  
22. $1.115 \times 10^{-4}$  
23. $9.9995 \times 10^{-1}$  
24. $3.18 \times 10^{-11}$  
25. $5.51 \times 10^{12}$  
26. $6.02 \times 10^5$
Chapter 4

Distributive Laws

We presented the properties of operations in Chapter 1, and we have reproduced them below for convenience.

**Theorem 21.** The following properties of operations hold.

1. **Addition of real numbers is commutative:** \( a + b = b + a \) for all real number \( a \) and \( b \).

2. **Addition of real numbers is associative:** \((a + b) + c = a + (b + c)\) for all real numbers \(a, b,\) and \(c\).

3. **Multiplication of real numbers is commutative:** \( a \cdot b = b \cdot a \) for all real number \(a\) and \(b\).

4. **Multiplication of real numbers is associative:** \((a \cdot b) \cdot c = a \cdot (b \cdot c)\) for all real numbers \(a, b,\) and \(c\).

5. **Multiplication distributes over addition:** \( a(b + c) = ab + ac \) for all real numbers \(a, b,\) and \(c\).

6. **There is an identity for addition:** if \( a \) is any real number, then \( a + 0 = 0 + a = a \).

7. **There is an identity for multiplication:** if \( a \) is any real number, then \( a \cdot 1 = 1 \cdot a = a \).

8. **Every real number has an additive inverse:** if \( a \) is a real number, then

\[
 a + (-a) = (-a) + a = 0.
\]

9. **Every real number except 0 has a multiplicative inverse:** if \( a \) is a real number, then

\[
 a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.
\]

In this chapter, we focus on consequences of the **distributive property**.
4.1 The Distributive Law

We will be using the distributive property in two ways, so we will give both forms here explicitly.

**Definition 4.1.1.** Let $a, b,$ and $c$ be any real numbers. Then

1. (Distributing)
   
   (a) $a(b + c) = ab + ac$
   
   (b) $(b + c)a = ba + ca$

2. (Factoring)

   (a) $ab + ac = a(b + c)$
   
   (b) $ba + ca = (b + c)a$.

Notice that the second form is really saying the same thing as the first form; we have just used the symmetric property of equality to rewrite it. When we apply the second form, we refer to the process as **factoring**.

Why should the distributive property work? Consider the following example.

**Example 4.1.1.** Applying the order of operations, we see that


Applying the distributive property, we see that

$$4(3 + 9) = 4 \cdot 3 + 4 \cdot 9 = 12 + 36 = 48.$$ 

The diagram below illustrates the distributive property for this example.

![Figure 4.1: 4(3 + 9) = 4 \cdot 3 + 4 \cdot 9](image)

The idea is that the total number of squares is the same, regardless of whether we count the first 9 columns separately from the last 3 columns, or whether we count all 12 columns together. In fact, figures like the one above motivate the distributive law in the first place.
Example 4.1.2.

\[(4x^2 + 3x)(2x) = 4x^2(2x) + 3x(2x)\]
\[= 8x^{2+1} + 6x^{1+1}\]
\[= 8x^3 + 6x^2,\]
\[\square\]

We often use the distributive law to combine like terms in a polynomial expression.

Example 4.1.3. Simplify \(4x + 7x\).

Solution: Notice that both terms contain a factor of \(x\), so we have an expression of the form in 2b above. Therefore,

\[4x + 7x = (4 + 7)x\]
\[= 11x.\]
\[\square\]

We use this idea to define addition of polynomial expressions.

**Definition 4.1.2.** If \(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0\) and \(b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0\) are polynomial expressions, then

\[(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0) + (b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0)\]
\[= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0).\]

That is, to add two polynomial expressions, we simply combine their like terms.

**Example 4.1.4.**

\[3x^2 + 5x + 4 + (4x^2 - 7x + 1) = (3 + 4)x^2 + (5 - 7)x + (4 + 1)\]
\[= 7x^2 - 2x + 5.\]
\[\square\]

Again, all we have really done is apply the Distributive Law 2b to the \(x^2, x,\) and constant terms. Also, the sign is part of the coefficient, so we wrote the coefficient of \(x\) as \((5 - 7)\); we could just as well have written it as \(5 + (-7)\).

**Example 4.1.5.**

\[6x^4 - 3x + 9 + (2x^2 - 7x + 3) = 6x^4 + 2x^2 + (-3 - 7)x + (9 + 3)\]
\[= 6x^4 + 2x^2 - 10x + 12.\]

Notice here that the second polynomial did not have an \(x^4\) term, so the coefficient 6 of the \(x^4\) term in the first polynomial was unchanged. Likewise, the coefficient 2 of \(x^2\) in the second polynomial was unchanged.
Example 4.1.6.

\[
\left( \frac{2}{5} x^2 - \frac{3}{4} x + 2 \right) + \left( \frac{3}{8} x^2 + \frac{2}{3} x - \frac{7}{4} \right) = \left( \frac{2}{5} + \frac{3}{8} \right) x^2 + \left( -\frac{3}{4} + \frac{2}{3} \right) x + \left( 2 - \frac{7}{4} \right) \\
= \frac{2 \cdot 8 + 3 \cdot 5}{5 \cdot 8} x^2 + \frac{-3 \cdot 3 + 2 \cdot 4}{4 \cdot 3} x + \frac{8 - 7}{4} \\
= \frac{31}{40} x^2 - \frac{1}{12} x + \frac{1}{4}.
\]

Combining like terms can be very tedious if you have to consciously employ the distributive law every time. You should practice combining like terms until you are comfortable computing

\[3x^3 + 7x^3 = 10x^3\]

rather than

\[3x^3 + 7x^3 = (3 + 7)x^3 = 10x^3.\]

However, by the same token, you should continue writing the middle step until you do become comfortable skipping it. Whether you write the middle step or not, though, you should always be aware that combining like terms makes use of the distributive law. In the rest of this book, we will sometimes write the middle step, and sometimes omit it.

The distributive laws can also be used to factor polynomial expressions.

Example 4.1.7. Factor 3x^2 + 5x.

Solution: Since each term has a factor of x, we may apply 2b from above. Thus,

\[3x^2 + 5x = 3x \cdot x + 5x = (3x + 5)x.\]

We cannot factor (in a meaningful way) any further since the remaining terms do not share a common factor. (They are not like terms.)

Example 4.1.8. Apply the distributive law to factor out a common factor from 4x^3−12x^2+8x.

Solution 1: The three terms all share a factor of 4x:

\[4x^3 - 12x^2 + 8x = 4x(x^2) - 4x(3x) + 4x(2) = 4x(x^2 - 3x + 2).\]

Solution 2: Strictly speaking, the three terms also share a factor of 8x:

\[4x^3 - 12x^2 + 8x = 8x \left( \frac{1}{2} x^2 \right) - 8x \left( \frac{3}{2} x \right) + 8x(1) = 8x \left( \frac{1}{2} x^2 - \frac{3}{2} x + 1 \right).\]
Typically, though, when we factor polynomials with integer coefficients, we also want the factors to have integer coefficients, so we would most likely prefer solution 1.

Both solutions share have something in common: the factor we looked for was the highest power of \( x \) occurring in the polynomial expression.

\[ 5x^8 - 11x^3 = 5x^5x^3 - 11x^3 = (5x^5 - 11)x^3. \]

Again, we found the highest power of \( x \) the terms shared.

Notice how in all of these examples we are using the Multiplication of Like Bases Theorem in reverse: when we see that \( x^8 \) and \( x^3 \) both have a factor of \( x^3 \), we rewrite \( x^8 \) as \( x^5x^3 \). Check that this is legitimate!

\[ \frac{x^2}{3x+1} - \frac{2x^2}{x-2} = \frac{1}{3x+1} \cdot x^2 - \frac{2}{x-2} \cdot x^2 = \left( \frac{1}{3x+1} - \frac{2}{x-2} \right) x^2. \]

The distributive laws apply to rational expressions as well as to polynomial expressions.

\[ x^{\frac{3}{2}} - 2x^{\frac{1}{2}} = x^{\frac{3}{2}}x^{\frac{1}{2}} - 2x^{\frac{1}{2}} = (x^{\frac{3}{2}} - 2)x^{\frac{1}{2}} = (x^2 - 2)x^{\frac{1}{2}}. \]

As before, we looked for the highest power of \( x \) the two terms shared; in this case, it was \( x^{\frac{1}{2}} \).

The important thing to remember from this is that the distributive law and factoring are two sides of the same coin; their difference is only a matter of perspective.

**Section 4.1 Exercises**

Expand each product by applying the distributive law 1a or 1b.
Combine like terms in each polynomial expression.

13. \(4x^2 + 5x + 2x^2 + 3x\)
14. \(3x^3 - 4x^2 + x - 4x^3 + x^2\)
15. \(-5x^4 + 2x^4\)
16. \(3x^2 + 11x^2\)
17. \(4x - 9x\)
18. \(2x^2 + 5x - 3x^2 + x\)
19. \(6x^7 + 4x^3 - 2x^7 + 3x\)

Factor each expression by applying the distributive law 2a or 2b.

25. \(2x^2 + 4x\)
26. \(3(x + 1) + x(x + 1)\)
27. \(4x^3 + 8x^2\)
28. \(\frac{x}{x^2 + 1} + \frac{x}{x^2 - 1}\)
29. \(\frac{1}{3}x - \frac{1}{3}x^2\)
30. \(x^4 - 5x^3\)

Add the polynomial expressions as indicated.

37. \((x^2 + 5x - 3) + (4x^2 + 3x + 2)\)
38. \((\frac{2}{3}x^4 + 5) + \left(\frac{1}{8}x^4 + x^2 + \frac{11}{3}\right)\)
39. \((5x + 7) + (3x^2 - x - 3)\)
40. \((8x^{12} - 4x^8 + 8x^4 - 4) + (3x^{12} + 2x^8 - x^4 + 4)\)
41. \((-2x^3 - 4x) + (5x^3 + x^2 + 7x)\)

42. \((8x^2 + 5) + (3x^2 + 4)\)

### 4.2 Generalizations of the Distributive Law

The Distributive Law states that multiplication distributes over a sum of two terms; in one form, this is

\[ a(b + c) = ab + ac. \]

However, there are several consequences of this that we will refer to as the Generalized Distributive Law.

**Example 4.2.1.**

\[
3(7x - 5) = 3(7x + (-5))
\]
\[
= 3(7x) + 3(-5)
\]
\[
= 21x + (-15)
\]
\[
= 21x - 15.
\]

This example illustrates the fact that multiplication distributes over subtraction as well as addition.

**Example 4.2.2.**

\[
3x^2(-2x^5 - 4x^2) = 3x^2(-2x^5) - (3x^2)(-4x^2)
\]
\[
= -6x^{2+5} - (-12x^{2+2})
\]
\[
= -6x^7 + 12x^4.
\]

**Example 4.2.3.**

\[
(3 + 2t + 4t^2)t = [(3 + 2t) + 4t^2]t \quad \text{Order of Operations}
\]
\[
= (3 + 2t)t + (4t^2)t \quad \text{Distributive Law 1b}
\]
\[
= 3t + (2t)t + 4t^2t \quad \text{Distributive Law 1b}
\]
\[
= 3t + 2t^2 + 4t^3 \quad \text{Multiplication of Like Bases}
\]

This example illustrates the fact that the distributive property can be applied to more than just two terms.

**Example 4.2.4.**

\[
5(-4x^3 + x^2 - 3x + 4) = 5(-4x^3) + 5x^2 + 5(-3x) + 5(4)
\]
\[
= -20x^3 + 5x^2 - 15x + 20.
\]
Example 4.2.5. Recall that if $a$ is a real number, then $-a = (-1)a$.

$-(3 + x) = (-1)(3 + x)$
$\quad = (-1)(3) + (-1)(x)$
$\quad = -3 + (-x)$
$\quad = -3 - x.$

That is, $-(3 + x) = -3 - x$, so the $-$ sign distributes over the addition, as well. The effect of the $-$ sign was to change the sign of each term in $3 + x$.

Example 4.2.6. We can exploit the above example to compute $7x - (3 + x)$.

$7x - (3 + x) = 7x + [- (3 + x)]$
$\quad = 7x + [-3 - x]$
$\quad = 7x - 3 - x$
$\quad = 7x - 3 - 1x$
$\quad = (7 - 1)x - 3$
$\quad = 6x - 3.$

Example 4.2.7.

$-(x - 2) = (-1)(x - 2)$
$\quad = (-1)x + (-1)(-2)$
$\quad = -x + 2.$

Again, the $-$ sign in front of the polynomial changed each sign in the polynomial.

Example 4.2.8.

$-(4x^3 - 3x^2 - 5x + 8) = (-1)(4x^3 - 3x^2 - 5x + 8)$
$\quad = (-1)(4x^3) + (-1)(-3x^2) + (-1)(-5x) + (-1)(8)$
$\quad = -4x^3 + 3x^2 + 5x - 8.$

Once more, the effect of the $-$ sign was to change the sign of every term in the polynomial.
Example 4.2.9.

\[-3(2x^2 - 4x + 3) = (-3)(2x^2) + (-3)(-4x) + (-3)(3)\]
\[= -6x^2 + 12x - 9.\]

These examples show us how to subtract polynomial expressions.

Example 4.2.10.

\[(5x^2 - 3x - 1) - (x^2 + 4x - 6) = (5x^2 - 3x - 1) + (-x^2 - 4x + 6)\]
\[= 5x^2 - 3x - 1 - 1x^2 - 4x + 6\]
\[= (5 - 1)x^2 + (-3 - 4)x + (-1 + 6)\]
\[= 4x^2 - 7x + 5.\]

That is, we just subtract each term individually.

Example 4.2.11.

\[(3t^4 + 6t^2 - 7t - 5) - (4t^3 + 3t^2 - 5t - 2) = 3t^4 + 6t^2 - 7t - 5 - 4t^3 - 3t^2 + 5t + 2\]
\[= 3t^4 - 4t^3 + (6 - 3)t^2 + (-7 + 5)t + (-5 + 2)\]
\[= 3t^4 - 4t^3 + 3t^2 - 2t - 3.\]

The theorem below summarizes our discoveries.

**Theorem 22** (Generalized Distributive Law). Let $a, b, \text{ and } c$ be real numbers. Then

1. $-(a + b) = -a - b$
2. $-(a - b) = -a + b$
3. $a(b - c) = ab - ac$
4. $(b - c)a = ba - ca$
5. $ab - ac = a(b - c)$
6. $ba - ca = (b - c)a$
7. $a(x_1 + x_2 + \ldots + x_n) = ax_1 + ax_2 + \ldots + ax_n$ for any real numbers $x_1, x_2, \ldots, x_n$.

**Section 4.2 Exercises**

Apply the Generalized Distributive Law to expand each expression.
1. \(- (x + 3)\)  
2. \(- (x^2 + 4x)\)  
3. \(- (3x - 5)\)  
4. \(4(12x - 5)\)  
5. \(- x^2(x^3 - 4x)\)  
6. \(- \frac{3}{4}(4x - 12)\)  
7. \((7x - 4)x\)  
8. \((2x^2 - 5x)(4x^2)\)  
9. \((5t - 15) \cdot \frac{1}{5}\)  
10. \(-3x(x^2 + 3x - 4)\)

Apply the generalized distributive law to factor each expression.

11. \(\frac{3}{4}x^3 \left(\frac{1}{3}x^2 + \frac{2}{3}x + \frac{1}{3}\right)\)
12. \(\left(x^\frac{4}{3} - 3x^\frac{2}{3}\right) x^\frac{1}{3}\)
13. \(x^\frac{3}{2} \left(3x^\frac{1}{2} - 7x^\frac{1}{4}\right)\)
14. \(- (x^6 - 4x^2 - 1)\)
15. \(4x(x^3 - 7x^2 + 3x + 2)\)
16. \(-12x^4(-x^2 + 3x)\)
17. \(-(5x^3 + x^2 + 1)x^3\)
18. \(-3(x^4 + 2x^2 + 1)x^4\)

Add or subtract the polynomial expressions as indicated.

19. \(x^2 - x\)
20. \(4x^3 - 6x\)
21. \(-x^4 - 4x^3\)
22. \(-3x^2 + 12\)
23. \(9x^3 + 12x^2 + 21x\)
24. \(8 - 10x^2\)
25. \(-w - 1\)
26. \(-\frac{7}{8}x^4 - \frac{1}{8}x^2\)
27. \(-2x - 2\)
28. \(-12x + 16\)
29. \(-x^\frac{3}{2} - x^\frac{5}{3}\)
30. \(x^2\sqrt{x} - 5\sqrt{x}\)
31. \(12\sqrt{x + 1} - 9\sqrt{x + 1}\)
32. \(\frac{6x}{\sqrt{x}} - \frac{3x^2}{\sqrt{x + 1}}\)
33. \(-\sqrt{x^2 + 4} - 6x\sqrt{x^2 + 4}\)
34. \(28x^9 - 21x^{14}\)
35. \(3x^2\sqrt{x} - 5x\sqrt{x} + 4\sqrt{x}\)
36. \(80x^3 + 40x^2 + 20x + 10\)
5. \((x^6 - 7x^3 + 5x) - (2x^6 - 3x^3 + 5x)\)
6. \((12x^3 + 5x + 1) - (3x^3 + 2x^2 + 4x + 3)\)
7. \((4x^3 + 7x - 3) + (9x^2 - 5x + 1)\)
8. \(\left(\frac{3}{7}x + 2\right) - \left(\frac{5}{7}x + \frac{4}{5}\right)\)
9. \((7x^3 - 3x^2 + 4) - (2x^3 - 3x^2 + 4)\)
10. \(\left(-\frac{7}{9}x^4 + \frac{4}{9}x^2 + \frac{2}{9}\right) - \left(\frac{1}{3}x^4 - \frac{2}{3}x^2 + \frac{4}{9}\right)\)

### 4.3 Multiplication of Polynomial Expressions

We can also apply the distributive law when both factors have more than one term.

**Example 4.3.1**. Expand \((3 + 4)(6 + 2)\).

**Solution**: We will begin with the Distributive Law 1a. That is, we will think of \(3 + 4\) as a single quantity.

\[
(3 + 4)(6 + 2) = (3 + 4)6 + (3 + 4)2 \quad \text{Distributive Law 1a}
\]
\[
= (3 \cdot 6 + 4 \cdot 6) + (3 \cdot 2 + 4 \cdot 2) \quad \text{Distributive Law 1b}.
\]

Notice that each term in the first factor was multiplied by each term in the second factor, and those products were all added together.

The figure below illustrates this computation.

![Figure 4.2: \((3 + 4)(6 + 2) = 3 \cdot 6 + 4 \cdot 6 + 3 \cdot 2 + 4 \cdot 2\)](image)

The idea is that the total number of squares is the same, regardless of how we count them.

The same principle applies even when the expressions contain variables.
Example 4.3.2.

\[(x + 2)(x + 3) = (x + 2)x + (x + 2)3\]  
Distributive Law 1a
\[= x \cdot x + 2x + x \cdot 3 + 2 \cdot 3\]  
Distributive Law 1b
\[= x^2 + 2x + 3x + 6\]  
Commutativity of Multiplication
\[= x^2 + (2 + 3)x + 6\]  
Distributive Law 2b
\[= x^2 + 5x + 6\]  
\[2 + 3 = 5.\]

□

Notice the application of Distributive Law 2b to combine like terms.

Example 4.3.3.

\[(x^2 + 1)(x^3 - 2) = (x^2 + 1)x^3 - (x^2 + 1) \cdot 2\]  
Generalized Distributive Law
\[= x^2x^3 + 1x^3 - (x^2 \cdot 2 + 1 \cdot 2)\]  
Generalized Distributive Law.
\[= x^5 + x^3 - 2x^2 - 2\]  
Generalized Distributive Law

Again, notice that each term in the first factor was multiplied by every term in the second factor. This will always occur.

□

Example 4.3.4.

\[(x + 3)(x + 6) = (x + 3)x + (x + 3)6\]  
\[= x \cdot x + 3x + x \cdot 6 + 3 \cdot 6\]  
\[= x^2 + 3x + 6x + 18\]  
\[= x^2 + (3 + 6)x + 18\]  
\[= x^2 + 9x + 18.\]

□

Example 4.3.5.

\[(2x + 3)(x^2 - 3x + 4) = 2x(x^2 - 3x + 4) + 3(x^2 - 3x + 4)\]  
\[= 2x^3 - 6x^2 + 8x + 3x^2 - 9x + 12\]  
\[= 2x^3 + (-6 + 3)x^2 + (8 - 9)x + 12\]  
\[= 2x^3 - 3x^2 - x + 12.\]

□ Once more, you can see on the second line that every term in the first factor \((2x + 3)\) is multiplied by every term in the second factor \((x^2 - 3x + 4)\). We can take advantage of this observation to streamline our work.
Example 4.3.6.

\[(−4x - 5)(3x^2 - 6x - 4) = (−4x)(3x^2) + (−4x)(−6x) + (−4x)(−4)
+ (−5)(3x^2) + (−5)(−6x) + (−5)(−4)
= −12x^3 + 24x^2 + 16x − 15x^2 + 30x + 20
= −12x^3 + (24 − 15)x^2 + (16 + 30)x + 20
= −12x^3 + 9x^2 + 46x + 20.\]

Notice also that in each example we apply the definition for addition of polynomials to combine like terms.

Before we go on, let’s record the fact that each term of a factor is multiplied by every term of the other factor.

**Theorem 23.** Let \(a, b, c,\) and \(d\) be real numbers. Then

\[(a + b)(c + d) = ac + ad + bc + bd.\]

You may have heard this referred to as “FOIL” in another algebra class; we will not refer to it that way in this book. You are far better off learning the underlying principles (like the distributive law) instead of “clever” names that hide the mathematics.

**Example 4.3.7.** Let’s be a little more general now. Let \(c\) and \(d\) be any real numbers. In the product \((x + c)(x + d)\), we know that each term in the \(x + c\) factor will be multiplied by each term in the \(x + d\) factor, so we have

\[(x + c)(x + d) = x \cdot x + x \cdot xd + cx + cd
= x^2 + dx + cx + cd
= x^2 + (d + c)x + cd
= x^2 + (c + d)x + cd.\]

That is,

\[(x + c)(x + d) = x^2 + (c + d)x + cd.\]

Notice that in the product, the coefficient of \(x\) is the sum of the two constant terms from the factors, and the constant term of the product is the product of the constant terms.

What’s nice about this is that it doesn’t matter what \(c\) and \(d\) are! Because we used the symbols \(c\) and \(d\) instead of specific numbers, when we have specific numbers we can just substitute them for \(c\) and \(d\).

**Example 4.3.8.** Compute \((x + 2)(x + 3)\).

**Solution:** We have \(c = 2\) and \(d = 3\) from the last example. Thus,

\[(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \cdot 3
= x^2 + 5x + 6.\]
Notice how much less work this was than what we did in Example 4.3.2 above. It’s the same problem, and it’s still a consequence of the Distributive Law, but the work we did in Example 4.3.7 made it much easier to carry out this computation. That is part of the value of introducing symbols like $c$ and $d$ instead of always dealing with specific numbers.

Example 4.3.9.

$$(x + 4)(x - 9) = x^2 + [4 + (-9)]x + 4(-9)$$
$$= x^2 - 5x - 36.$$

Notice again that the $-$ sign is part of the constant term in $x - 9$.

Example 4.3.10. We can also use Theorem 4.3.1 to multiply more complicated-looking binomials. For example, let’s compute $(2x + 3)(3x + 7)$. We have $a = 2x, b = 3, c = 3x, \text{ and } d = 7$. Thus

$$(2x + 3)(3x + 7) = (2x)(3x) + (2x)(7) + (3)(3x) + (3)(7)$$
$$= 2 \cdot 3x^2 + (2 \cdot 7 + 3 \cdot 3)x + 3 \cdot 7$$
$$= 6x^2 + 23x + 21.$$

The second line may seem a little strange at first, but we included it to illustrate what is happening; there is a pattern at work here. Let’s consider the result term by term.

1. For the $x^2$ term in the product, we have only the product of the coefficients of the $x$ terms of each factor, $2 \cdot 3 = 6$.

2. For the $x$ term in the product, we have the sum of two products, $2 \cdot 7 + 3 \cdot 3$. This came from the coefficient of $x$ in each factor times the constant term of the other factor. Notice that what this amounts to is just looking at the products that will give the correct total degree, in this case, 1. (An $x$ term times a constant term gives another $x$ term.)

3. For the constant term of the product, we have only the product of the constant terms of each factor, $3 \cdot 7 = 21$.

Example 4.3.11. What if we have four unknown coefficients, $a, b, c,$ and $d$? Let’s find out!

$$(ax + c)(bx + d) = (ax)(bx) + (ax)(d) + (c)(bx) + (c)(d)$$
$$= abx^2 + adx + cbx + cd$$
$$= abx^2 + (ad + cb)x + cd$$
$$= abx^2 + (ad + bc)x + cd.$$
This allows us to quickly compute the product of any two binomials, so we will record it as a theorem.

**Theorem 24.** Let \( a, b, c, \) and \( d \) be real numbers. Then

\[
(ax + c)(bx + d) = abx^2 + (ad + bc)x + cd.
\]

We claimed that this will allow us to compute the product of any two binomials, but we already had a theorem for some binomials; namely, \((x + c)(x + d) = x^2 + (c + d)x + cd\). These two should give us the same result for \((x + c)(x + d)\), so let’s check. We have \(a = 1\) and \(b = 1\), so

\[
(ax + c)(bx + d) = (1 \cdot 1)x^2 + (1 \cdot c + 1 \cdot d)x + cd
\]

\[
= x^2 + (c + d)x + cd,
\]

which is the same result we found in Theorem 4.3.7. What a relief! Whenever you learn a more general result, you should compare it to special cases to make sure it agrees.

Let’s use the theorem in an example.

**Example 4.3.12.** Compute \((3x - 4)(2x + 1)\).

**Solution:** We have \(a = 3, b = 2, c = -4,\) and \(d = 1\). Thus

\[
(3x - 4)(2x + 1) = (3 \cdot 2)x^2 + (3 \cdot 1 + (-4) \cdot 2)x + (-4) \cdot 1
\]

\[
= 6x^2 - 5x - 4.
\]

Wasn’t that easy?

For convenience, we gather here the main results of this section. These are well worth memorizing, as that will greatly increase your proficiency. Here’s a study tip: the more problems you work, the less time you will have to spend actually “memorizing” these properties. They will embed themselves in your head if you work them enough.

**Theorem 25.** Let \( a, b, c, \) and \( d \) be real numbers. Then

1. \((a + b)(c + d) = ac + ad + bc + bd\)
2. \((x + c)(x + d) = x^2 + (c + d)x + cd\)
3. \((ax + c)(bx + d) = abx^2 + (ad + bc)x + cd\)

**Section 4.3 Exercises**

Draw a diagram like that of Figure 4.2 to illustrate each computation.
1. \((2 + 1)(3 + 2)\)  
3. \((1 + 4)(3 + 5)\)

2. \((3 + 3)(2 + 4)\)  
4. \((2 + 4)(4 + 5)\)

Multiply as indicated and simplify by combining like terms.

5. \((x + 9)(x - 4)\)  
19. \((x + 2)(3x - 1)\)  
34. \((3x + 5)(3x + 5)\)

6. \((x + 3)(x + 1)\)  
20. \((4x - 1)(4x - 3)\)  
35. \((x + \frac{1}{4})(x + \frac{1}{4})\)

7. \((x + 4)(x + 6)\)  
21. \((x - 1)(x + 1)\)  
36. \((2x + 2)(2x + 2)\)

8. \((x - 4)(x + 2)\)  
22. \((x - 3)(x + 3)\)  
37. \((5x - 7)(5x - 7)\)

9. \((x - 6)(x - 2)\)  
23. \((x + 8)(x - 8)\)  
38. \((x + a)(x - a)\)

10. \((x + 7)(x + 5)\)  
24. \((x - 12)(x + 12)\)  
39. \((x + 2)(x^2 - 3x + 1)\)

11. \((t + 2)(t - 4)\)  
25. \((3x + 2)(3x - 2)\)  
40. \((x - 4)(5x^2 + 8x - 3)\)

12. \((x - 8)(x - 10)\)  
26. \((5x + 6)(5x - 6)\)  
41. \((x^2 + 1)(x^3 - x - 2)\)

13. \((x - \frac{1}{2})(x + \frac{3}{4})\)  
27. \((11x + 4)(11x - 4)\)  
42. \((3x^2 + x)(2x^2 - 5x - 5)\)

14. \((x + \frac{3}{5})(x - \frac{4}{5})\)  
28. \((6x + 2)(6x - 2)\)  
43. \((2x^2 + 4x + 3)(3x^2 + 5x + 7)\)

15. \((3x + 1)(2x - 4)\)  
29. \((x + a)(x - a)\)  
44. \((x^2 + 4x + 2)(3x + 5)\)

16. \((2x - 5)(4x + 2)\)  
30. \((x + 2)(x + 2)\)  
45. \((-3x^2 - 5x - 11)(7x^2 - 2x - 3)\)

17. \((8x - 11)(4x + 3)\)  
31. \((x + 5)(x + 5)\)  
46. \((x^2 + 1)(x^2 - 2)\)

18. \((5x + 8)(3x - 6)\)  
32. \((x - 8)(x - 8)\)  
47. \((x^2 + 1)(x^2 - 2)\)

19. \((x + 2)(3x - 1)\)  
34. \((3x + 5)(3x + 5)\)

4.4 Special Forms

In this section we consider several products that have special, easily recognized forms. Learning these well can save you time and effort.

First, we have the perfect square form.

**Example 4.4.1.** Compute \((x + 3)^2\).

**Solution:** We know that \((x + 3)^2 = (x + 3)(x + 3)\), so we may compute as in the previous section.

\[
(x + 3)(x + 3) = x^2 + (3 + 3)x + 3 \cdot 3 = x^2 + 6x + 9.
\]
What makes this form easy to work with is that the coefficient of \(x\) is just twice the constant term of a factor, and the constant term is the square of the constant term of the factor.

**Example 4.4.2.** Compute \((x + a)^2\).

**Solution:**

\[
(x + a)^2 = (x + a)(x + a) = x^2 + (a + a)x + a \cdot a = x^2 + 2ax + a^2.
\]

This merits a theorem.

**Theorem 26.** Let \(a\) be a real number. Then

\[
(x + a)^2 = x^2 + 2ax + a^2.
\]

This allows us to compute such squares with great ease.

**Example 4.4.3.**

\[
(x + 7)^2 = x^2 + 2 \cdot 7x + 7^2 = x^2 + 14x + 49.
\]

**Example 4.4.4.**

\[
(x - 4)^2 = x^2 + 2(-4)x + (-4)^2 = x^2 - 8x + 16.
\]

**BEWARE:** Students are often tempted to just square each term of the factor and add those. However, carefully using the distributive law shows us that there is also an \(x\)-term that must not be neglected. **Don’t make the mistake of ignoring it!**

**Example 4.4.5.** What if we have \((3x + 4)^2\)? We can still compute as in the previous section.

\[
(3x + 4)^2 = (3x + 4)(3x + 4) = 3 \cdot 3x^2 + (3 \cdot 4 + 4 \cdot 3)x + 4 \cdot 4 = 9x^2 + 24x + 16.
\]
Notice that the coefficient of \(x^2\) in the product is the square of the coefficient of \(x\) in the factor, or \(3^2\). The coefficient of \(x\) in the product is twice the product of the coefficients in the factor, or \(2(3 \cdot 4)\). The constant term in the product is the square of the constant term in the factor, \(4^2 = 16\).

**Example 4.4.6.** Compute \((a + b)^2\).

**Solution:**

\[
(a + b)^2 = (a + b)(a + b) = (a \cdot a) + (ab + ba) + b \cdot b = a^2 + 2ab + b^2.
\]

This is general enough to make a theorem.

**Theorem 27 (Perfect Squares).** Let \(a\) and \(b\) be real numbers. Then

\[
(a + b)^2 = a^2 + 2ab + b^2.
\]

Again, this makes it easy to compute squares of binomials.

**Example 4.4.7.** For \((5x - 2)^2\), we have \(a = 5x\) and \(b = -2\).

\[
(5x - 2)^2 = (5x)^2 + 2(5x)(-2) + (-2)^2 = 25x^2 - 20x + 4.
\]

Another special form is the **difference of squares** form.

**Example 4.4.8.** Compute \((x + 3)(x - 3)\).

**Solution:**

\[
(x + 3)(x - 3) = x^2 + [3 + (-3)]x + (3)(-3) = x^2 + 0x - 9 = x^2 - 9.
\]

Notice that there is no \(x\) term in this; that makes it different from the perfect square form. Also, we simply have the \(x\) squared minus the 3 squared, a difference of squares.

**Example 4.4.9.** Compute \((4x + 2)(4x - 2)\).

**Solution:**

\[
(4x + 2)(4x - 2) = 4 \cdot 4x^2 + [4(-2) + 2(4)]x + (2)(-2) = 16x^2 + 0x - 4 = 16x^2 - 4.
\]

Again, we have no \(x\) term, and what we do have is \((4x)^2 - 2^2\), a difference of squares.
Example 4.4.10. Compute \((a + b)(a - b)\).

Solution:

\[
(a + b)(a - b) = a \cdot a + a(-b) + b(a) + b(-b)
= a^2 - ab + ab - b^2
= a^2 - b^2.
\]

This gives us the following theorem.

**Theorem 28** (Difference of Squares). Let \(a\) and \(b\) be real numbers. Then

\[
(a + b)(a - b) = a^2 - b^2.
\]

This works regardless of whether \(a\) and \(b\) include \(x\)’s or not.

Example 4.4.11. Compute \((3x - 7)(3x + 7)\).

Solution:

\[
(3x - 7)(3x + 7) = (3x)^2 - 7^2
= 3^2x^2 - 49
= 9x^2 - 49.
\]

Example 4.4.12.

\[
(4x + 2)(4x - 2) = (4x)^2 - 2^2
= 16x^2 - 4.
\]

Example 4.4.13. The difference of squares form can also make it easier to multiply some large numbers in your head. For example,

\[
32 \cdot 28 = (30 + 2)(30 - 2)
= 30^2 - 2^2
= 900 - 4
= 896.
\]

The trick is to find a number, like 30, that you can square in your head easily, but this isn’t always possible.
Example 4.4.14.

\[ 76 \cdot 84 = (80 - 4)(80 + 4) \]
\[ = 80^2 - 4^2 \]
\[ = 6400 - 16 \]
\[ = 6384. \]

For convenience, we list the major results of this section here.

**Theorem 29.** Let \(a, b, c,\) and \(d\) be real numbers. Then

1. \((a + b)^2 = a^2 + 2ab + b^2\) (Perfect square)
2. \((a + b)(a - b) = a^2 - b^2\) (Difference of squares)

**Section 4.4 Exercises**

Use the results of this section to compute.

1. \((x + 1)^2\)
2. \((x + 5)^2\)
3. \((t + 2)^2\)
4. \((5y + 2)^2\)
5. \((3x - 1)^2\)
6. \(\left(\frac{1}{x} + 2\right)^2\)
7. \((\sqrt{x} + 1)^2\)
8. \((x + 1)(x - 1)\)
9. \((x - 6)(x + 6)\)
10. \((x + 12)(x - 12)\)
11. \((x - \sqrt{2})(x + \sqrt{2})\)
12. \(\left(\frac{1}{x} + 3\right)\left(\frac{1}{x} - 3\right)\)
13. \((3x + 2)(3x - 2)\)
14. \((4x + 9)(4x - 9)\)
15. \((7x - 5)(7x + 5)\)
16. \((2x + 1)(2x - 1)\)
17. \(\left(\frac{x}{3} + 4\right)^2\)
18. \(\left(\frac{x + 2}{5}\right)^2\)
19. \((\sqrt{x} + 2)^2\)
20. \((\sqrt{3} - 1)^2\)
21. \((x^2 + 1)(x^2 - 1)\)
22. \((t + u)(t - u)\)
23. \((\sqrt{x} - 1)(\sqrt{x} + 1)\)

Each expression below is a perfect square trinomial. Determine what binomial was squared to attain the given trinomial.

25. \(x^2 + 4x + 4\)
26. \(x^2 - 4x + 4\)
27. \(x^2 + 6x + 9\)
28. \(x^2 - 10x + 25\)
29. \(4x^2 + 20x + 25\)
30. \(x^2 - 18x + 81\)
Each expression below is a difference of squares. Determine which two binomials were multiplied to attain the given difference of squares.

31. $x^2 - 1$  
32. $x^2 - 4$  
33. $x^2 - 16$  
34. $x^2 - 49$  
35. $4x^2 - 81$  
36. $25x^2 - 36$

### 4.5 Factoring Polynomials with the Distributive Law

Recall that we also used the distributive law to factor. We can do the same with polynomials; this is an aid to solving equations. In general, factoring is more difficult than multiplying, so we need to develop some tools to help us. We will do this by revisiting the results of the last two sections. We write them in a slightly different form to make them more useful to us now.

**Theorem 30.** Let $a, b, c,$ and $d$ be real numbers. Then

1. $ac + ad + bc + bd = (a + b)(c + d)$
2. $x^2 + (a + b)x + ab = (x + a)(x + b)$
3. $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
4. $a^2 + 2ab + b^2 = (a + b)^2$ (Perfect square)
5. $a^2 - b^2 = (a + b)(a - b)$ (Difference of squares)

We have just used the fact that equality is symmetric to reverse the equations. Now let’s see how these help us factor.

**Example 4.5.1.** Factor $x^2 - 4$.

**Solution:** We can recognize that this is a difference of squares;

$$x^2 - 4 = x^2 - 2^2.$$

From 5 above, we know that we can factor this as

$$x^2 - 2^2 = (x + 2)(x - 2).$$

**Example 4.5.2.** Factor $16x^2 - 25$.

**Solution:** We again have a difference of squares; this time, it is

$$(4x)^2 - 5^2.$$  

Again using 5 above, we factor this as

$$(4x - 5)(4x + 5).$$
The difference of squares form is usually an easy one to recognize. Perhaps the next simplest is the perfect square form.

**Example 4.5.3.** Factor \(x^2 + 10x + 25\).

**Solution:** We can rewrite this as \(x^2 + 2 \cdot 5x + 5^2\), so we have a perfect square. Thus,
\[x^2 + 10x + 25 = (x + 5)^2.\]

**Example 4.5.4.** Factor \(x^2 + 12x + 36\).

**Solution:** We can rewrite this as \(x^2 + 2 \cdot 6x + 6^2\), so we have a perfect square. Thus,
\[x^2 + 12x + 36 = (x + 6)^2.\]

**Example 4.5.5.** Factor \(9x^2 + 24x + 16\).

**Solution:** We can rewrite this as \((3x)^2 + 2(3 \cdot 4)x + 4^2\), so we have a perfect square. (See 4 above.) Thus,
\[9x^2 + 24x + 16 = (3x + 4)^2.\]

More general forms can be more difficult to factor, but we can use part 2 of Theorem 4.5.1 to detect a pattern. We have that
\[x^2 + (a + b)x + ab = (x + a)(x + b).\]

Thus, in order to factor such a trinomial, we need to find a factor pair \(a\) and \(b\) of the constant term \(ab\) whose sum is the coefficient of \(x\) \((a + b)\). (Study the equation until that is clear to you. Remember, \(ab\) is the product of \(a\) and \(b\), and \(a + b\) is the sum of \(a\) and \(b\).)

**Example 4.5.6.** Factor \(x^2 - 4x + 3\).

**Solution:** The factor pairs of 3 are 1 and 3 and \(-1\) and \(-3\). The two whose sum is \(-4\) (the coefficient of \(x\)) are \(-1\) and \(-3\), so we rewrite \(x^2 - 4x + 3\) as
\[
x^2 - 4x + 3 = x^2 + (-1 + (-3))x + (-1)(-3) = (x + (-1))(x + (-3)) = (x - 1)(x - 3).
\]

Since 3 has only the four factors shown, it wasn’t too difficult to determine which added up to \(-4\). Let’s try one in which the constant term has more factors.
Example 4.5.7. Factor $x^2 + 8x + 12$.

Solution: The factor pairs of 12 are 1 and 12, −1 and −12, 2 and 6, −2 and −6, 3 and 4, and −3 and −4. Examining these shows us that 2 and 6 are the factors whose sum is 8 (the coefficient of $x$). Thus

$$x^2 + 8x + 12 = x^2 + (2 + 6)x + (2)(6) = (x + 2)(x + 6) \quad \text{Theorem 4.5.1, part 2.}$$

□

With practice, finding the appropriate factors becomes much easier.

Before we proceed to more complicated factors, let’s see how the principles we’ve seen so far apply to our special forms.

Example 4.5.8. Factor $x^2 − 9$.

Solution 1: We can see that this is a difference of squares form, so we factor it as $(x + 3)(x - 3)$.

Solution 2: Suppose we didn’t recognize $x^2 − 9$ as a difference of squares. Part 2 of Theorem 4.5.1 tells us to find factor pairs of −9 that add up to 0 (the coefficient of $x$). The factor pairs of −9 are 1 and −9, −1 and 9, and 3 and −3. Since $3 + (-3) = 0$, we have

$$x^2 − 9 = x^2 + (3 + (-3))x + (3)(-3) = (x + 3)(x + (-3)) \quad \text{Theorem 4.5.1, part 2} \quad \text{□}$$

Thus, our principle applies to difference of squares forms, too! Let’s see how it fairs for perfect square forms.

Example 4.5.9. Factor $x^2 + 6x + 9$.

Solution 1: We can recognize $x^2 + 6x + 9$ as a perfect square trinomial that factors as $(x + 3)^2$.

Solution 2: Suppose we didn’t recognize $x^2 + 6x + 9$ as a perfect square trinomial. We need a factor pair of 9 whose sum is 6 (the coefficient of $x$). The factor pairs of 9 are 1 and 9 and 3 and 3. The sum of 3 and 3 is 6, so this is the pair we want. We have

$$x^2 + 6x + 9 = x^2 + (3 + 3)x + (3)(3) = (x + 3)(x + 3) \quad \text{Theorem 4.5.1, part 2.} \quad \text{□}$$

Our principle applies to perfect square trinomials, as well!

Example 4.5.10. Factor $x^2 + 2x + 2$.

Solution: The factor pairs of 2 are 1 and 2 and −1 and −2. Since neither pair adds up to 2 (the coefficient of $x$), we’re stuck! We do not have the techniques to factor this trinomial.
This last example illustrates an important point: Theorem 4.5.1 does not cover all of the possibilities! For some polynomials, we will just have to wait for a later course.

Let’s move on to some more sophisticated examples. Consider part 3 of Theorem 4.5.1. The coefficient of $x^2$ is $ac$, and the constant term is $bd$; if we multiply these together, we get $acbd$. This new number has factors $ad$ and $bc$ whose sum is the coefficient of $x$, and this is the idea we will exploit. We’ll make this more concrete with an example.

**Example 4.5.11.** Factor $2x^2 + 7x + 3$.

**Solution:** Since the coefficient of $x^2$ isn’t 1, we can’t use part 2 of Theorem 4.5.1, so we will have to use part 3. Unfortunately, part 3 of Theorem 4.5.1 is too difficult to apply directly, so we will come at it from another direction.

The product of 2 and 3 is 6, so we need factors of 6 that add up to 7. The factors 1 and 6 add up to 7, so we will rewrite our polynomial:

$$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3.$$  

Notice how we split the $7x$ term into two new terms, $6x$ and $x$. Now we will factor this new expression by grouping terms in a convenient way.

$$2x^2 + 6x + x + 3 = (2x^2 + 6x) + (x + 3) = 2x(x + 3) + 1(x + 3) \text{ Distributive Law} = (2x + 1)(x + 3) \text{ Distributive Law}.$$  

Aha! The Distributive Law comes to the rescue again!

**Example 4.5.12.** Factor $4x^2 + 4x - 3$.

**Solution 1:** We again use the idea behind part 3 of Theorem 4.5.1. Since $4(-3) = -12$, we need a factor pair of $-12$ whose sum is 4 (the coefficient of $x$). The factor pairs are 1 and $-12$, $-1$ and 12, 2 and $-6$, $-2$ and 6, 3 and $-4$, and $-3$ and 4. The pair whose sum is 4 is $-2$ and 6. Thus, we have

$$4x^2 + 4x + 3 = 4x^2 - 2x + 6x - 3 = (4x^2 - 2x) + (6x - 3) = 2x(2x - 1) + 3(2x - 1) \text{ Distributive Law} = (2x + 3)(2x - 1) \text{ Distributive Law}.$$  

**Solution 2:** You may be wondering what happens if you break $4x$ into $6x - 2x$ instead of $-2x + 6x$. Let’s see:

$$4x^2 + 4x + 3 = 4x^2 + 6x - 2x - 3 = (4x^2 + 6x) + (-2x - 3) = 2x(2x + 3) - (2x + 3) \text{ Distributive Law} = (2x - 1)(2x + 3) \text{ Distributive Law}.$$  

We end up with the same factors, just written in the opposite order.
Example 4.5.13. Factor $16x^2 + 14x + 3$.

Solution: We need a factor pair of $16 \cdot 3 = 48$ whose sum is 14. We can ignore the factor pairs with $-$ signs since everything in this example is positive. The factor pairs are 1 and 48, 2 and 24, 3 and 16, 4 and 12, and 6 and 8. Since $6 + 8 = 14$, this is the pair we want. Now we have

$$16x^2 + 14x + 3 = 16x^2 + 8x + 6x + 3$$
$$= (16x^2 + 8x) + (6x + 3)$$
$$= 8x(2x + 1) + 3(2x + 1)$$
$$= (8x + 3)(2x + 1)$$

Distributive Law

Distributive Law

Work all of the exercises. Factorization becomes easier as you gain experience, and you only gain experience through practice!

Section 4.5 Exercises

Identify each expression as a perfect square trinomial (PST), difference of squares (DOS), or neither (Neither). Then factor using the methods of this section, if possible. If the methods of this section fail, write “does not factor.”

1. $x^2 + 5x + 4$
2. $x^2 + 4x + 4$
3. $x^2 + 16x + 64$
4. $x^2 - 16$
5. $x^2 - 8x + 16$
6. $x^2 - 8x - 9$
7. $x^2 + 8x + 4$
8. $x^2 + 6x - 9$
9. $x^2 + 3x - 4$
10. $x^2 + 11x + 24$
11. $x^2 + 7x + 10$
12. $x^2 - 9x - 36$
13. $x^2 - 121$
14. $x^2 - 40x + 400$
15. $x^2 + 20x + 64$
16. $x^2 - 14x + 39$
17. $x^2 - 3x - 40$
18. $x^2 + 2x - 80$
19. $9x^2 - 49$
20. $4x^2 + 1$
21. $16x^2 + 40x + 25$
22. $6x^2 - x - 1$
23. $10x^2 - 3x - 1$
24. $2x^2 + x - 6$
25. $21x^2 + 29x - 10$
26. $x^2 - 2x - 8$
27. $-8x^2 - 10x + 7$
28. $-3x^2 - 5x + 12$
29. $10x^2 + 9x + 2$
30. $10x^2 + 17x + 3$
31. $8x^2 - 2x - 3$
32. $4x^2 + 10x - 6$
33. $4x^2 + 9x + 5$
34. $-10x^2 - 61x + 26$
35. $18x^2 + 21x + 6$
36. $8x^2 + 29x + 15$
Chapter 5

Lines

5.1 Linear equations

We have already solved a few linear equations in this book, but in this section we will look at solving them more systematically.

An equation is a (mathematical) statement that two quantities are equal; for example,

\[ 4 = 9 - 5 \]

is an equation. A linear equation is an equation with a variable (like \( x \)) that only appears to the first power. (That is, the equation may not have \( \frac{1}{x}, x^{-2}, x^2, \sqrt{x}, x^{-4/3}, x(x + 2) \), etc. Notice that \( x(x + 2) = x^2 + 2x \), so even though it looks like \( x \) only appears to the first power, there is a hidden second power in there!)

Our goal when confronted with such an equation is to find out what value or values of \( x \) make it true. That is, we need to solve for the variable. The fundamental principle to keep in mind is that if we perform the same arithmetical operations on both sides of an equation, the result is still an equation. If we don’t it probably isn’t.

Example 5.1.1. Four months ago, James turned 21. How many months old is James?

Solution: For simplicity, let’s let \( a \) represent James’ present age in months (\( a \) for \( a \)ge).

Four months ago, his age was \( a - 4 \). Also, four months ago, James turned 21, which is \( 21 \cdot 12 = 252 \) months. Since both of these quantities represent James’ age four months ago, they must be equal:

\[ a - 4 = 252. \]

If we add 4 to both sides of this equation, we get

\[ a = 256, \]

so James is 256 months old.

\[ \square \]

Wait a minute! How did we get from \( a - 4 = 252 \) to \( a = 256 \)? Why did we do what we did? Why did it work? In the next several examples, we will explore the answers to these questions by trying to solve some linear equations without contexts.
Example 5.1.2. Suppose we have the equation $x - 8 = 4$. We want to know the value of $x$. Our approach is to perform operations on both sides of the equation so that the end result is of the form
\[ x = \text{(something with no } x\text{'s)}. \]
At that stage, the equation itself tells us the value of $x$ that makes it true. Let’s try this for the equation $x - 8 = 4$.

Remember that we want our equation to say $x = \text{(something with no } x\text{'s)}$. The equation we are given almost looks like this already, but on the left-hand side of the equal sign there is an extra “$-8$” that we don’t want. The simplest way to get rid of that $-8$ is to add 8 to both sides of the equation:
\[ (x - 8) + 8 = 4 + 8. \]
The left-hand side is the same as $[x + (-8)] + 8$, so we may use the fact that addition is associative to rewrite our equation as
\[ x + [(-8) + 8] = 4 + 8. \]
Now, we know that $-8 + 8 = 0$ and $4 + 8 = 12$, so let’s make these changes and see what we have.
\[ x + 0 = 12. \]
But this is wonderful! Since $x + 0 = x$, this just becomes $x = 12$, so the value of $x$ that solves this equation is $x = 12$. We can check that to make sure, too:
\[ (12) - 8 = 4, \]
just like we were hoping.

\[ \square \]

Ordinarily, we will not write out so many details. But there are a few things to notice in here: first, the reason we added 8 to both sides of the equation was so that we could turn the left-hand side into $x + 0$, which then turned into just $x$. It would have been perfectly legitimate to add 10 to both sides, but then the right-hand side would have been $x + 2$ instead of just $x$, so we wouldn’t have been any closer to a solution. In other words, we picked just the right number to add, namely 8, so that the $-8$ we didn’t want would be eliminated.

The second thing to notice is that we relied on the fact that addition of real numbers is associative. Without that special property of addition, we could not have solved this equation so easily, if at all!

Thirdly, notice that we did add the 8 to both sides of the equation. We did make other changes on just one side, like rewriting $x + 0$ as $x$ on the left-hand side, but those are really the same thing. However, adding 8 to a side changes the side, so you have to make sure you change the other side in exactly the same way.

Finally, notice that we checked our solution at the end. This is important! This is how we know with complete certainty that we didn’t make any mistakes. It only takes a couple of seconds, and it can keep you from having an error in your solution.
Example 5.1.3. Solve $x + 5 = -2$ for $x$.

**Solution:** This time, the left-hand side has a 5 we don’t want. The way to get rid of this is to add $-5$ to both sides of the equation.

$$x + 5 + (-5) = -2 + (-5).$$

Notice that this time we didn’t show the grouping of the $x + 5$ on the left-hand side. We are relying on the associativity of addition of real numbers here! Now our equation simplifies to

$$x + 0 = -7,$$

so $x = -7$ is our solution.

Let’s check again: $(-7) + 5 = -2$, which is what we wanted.

□

Example 5.1.4. Solve $3x - 2 = 2x + 1$ for $x$.

**Solution:** This time we have $x$’s on both sides of the equation. If we are going to end up with $x=$(something with no $x$’s), we need to get them on the same side. Let’s add $-2x$ to both sides of the equation so we can eliminate the $2x$ from the right-hand side.

\[
\begin{align*}
-2x + 3x - 2 &= -2x + 2x + 1 \quad \text{Adding } -2x \text{ to both sides} \\
(-2 + 3)x - 2 &= (-2 + 2)x + 1 \quad \text{Distributive Property} \\
(1)x - 2 &= (0)x + 1 \quad \text{Simplification} \\
x - 2 &= 0 + 1 \quad \text{Simplification} \\
x - 2 &= 1 \quad \text{Simplification.}
\end{align*}
\]

Now this looks like the ones we had before! We can finish by adding 2 to both sides:

\[
\begin{align*}
x - 2 + 2 &= 1 + 2 \\
x + 0 &= 3 \\
x &= 3.
\end{align*}
\]

Once more, let’s check our answer. The left-hand side is $3(3) - 2 = 9 - 2 = 7$, and the right-hand side is $2(3) + 1 = 6 + 1 = 7$. We’re right!

□

Let’s try one that’s a little more sophisticated.

Example 5.1.5. Solve $2x + 5 = 13$ for $x$.

**Solution:** This time the right-hand side has two operations in effect. By the order of operations, $x$ was first multiplied by 2, and then 5 was added to the result. To undo this, we will have to reverse those operations. First, let’s isolate the $2x$ by subtracting 5 from both sides.

\[
\begin{align*}
2x + 5 - 5 &= 13 - 5 \\
2x + 0 &= 8 \\
2x &= 8.
\end{align*}
\]
Now the left-hand side is almost what we want, but there is an extra factor of 2. How do we get rid of it? Let’s multiply both sides by $\frac{1}{2}$.

$$\frac{1}{2} \cdot (2x) = \frac{1}{2} \cdot 8 \quad (5.1)$$

On the left-hand side, we need to make use of the fact that multiplication of real numbers is associative (just as above we used the fact that addition of real numbers is associative. Then we can rewrite our equation as

$$(\frac{1}{2} \cdot 2) x = \frac{1}{2} \cdot 8.$$  

Since $\frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \cdot 8 = 4$, we may simplify this to

$$(1)x = 4.$$  

But now $(1)x = x$, so we have

$$x = 4$$

as our solution.

Let’s check: $2(4) + 5 = 8 + 5 = 13$, as expected.

\[\square\]

Just as in the first set of examples, there are things to notice here. Firstly, we relied on the associativity of multiplication of real numbers. Without, that we would have been stuck at $2x = 8$.

Secondly, we chose to multiply both sides of $2x = 8$ by $\frac{1}{2}$ so that we would get just a $(1)x$ or $x$ on the left-hand side. It would have been perfectly legitimate to multiply both sides by 3, but we would have ended up with $6x = 24$ and been no closer to a solution.

Thirdly, we multiplied both sides of the equation by $\frac{1}{2}$. This is important!

Finally, do be sure to check your answer.

Let’s recap our strategies. When we see an equation of the form $x + a = b$, the quickest way to solve is to add $-a$ to both sides of the equation. This turns the left-hand side into $x + a - a = x + 0 = x$, which is what we want on that side.

If we have an equation of the form $ax = b$ (with $a \neq 0$), the quickest way to solve is to multiply both sides of the equation by $\frac{1}{a}$. This turns the left-hand side into $\frac{1}{a} \cdot ax = 1x = x$, which is again what we want on that side.

In both cases, we choose an operation that will get us closer to just having $x$ on one side.

If the equation is more complicated, like $ax + b = c$, solve for a multiple of $x$ first. (Above, that meant solve for the $2x$ first.) Then the resulting equation will look like $dx = e$, which we know how to solve.

**NOTE:** There is certainly more than one way to solve $ax + b = c$; above we offered a suggestion that might keep the equation a little simpler as you solve, but you do not have to solve for $ax$ first. Generally speaking, experience will help you decide how to begin. Read our examples below and try to work every problem. With practice, you can solve a lot of linear equations in your head!
Example 5.1.6. Solve $\frac{2}{3}x - 3 = \frac{5}{2}$ for $x$.

Solution:

\[
\frac{2}{3}x - 3 = \frac{5}{2} \quad \text{Original equation}
\]

\[
\frac{2}{3}x - 3 + 3 = \frac{5}{2} + 3 \quad \text{Add 3 to both sides}
\]

\[
\frac{2}{3}x + 0 = \frac{5}{2} + \frac{6}{2} \quad \text{Simplify and find a common denominator}
\]

\[
\frac{2}{3}x = \frac{11}{2} \quad \text{Simplify}
\]

\[
\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot \frac{11}{2} \quad \text{Multiply both sides by } \frac{3}{2}
\]

\[
1 \cdot x = \frac{33}{4} \quad \text{Simplify}
\]

\[
x = \frac{33}{4} \quad \text{Simplify}
\]

Thus $x = \frac{33}{4}$. Let’s check:

\[
\frac{2}{3} \cdot \frac{33}{4} - 3 = \frac{66}{12} - 3
\]

\[
= \frac{66}{12} - \frac{36}{12}
\]

\[
= \frac{30}{12}
\]

\[
= \frac{12}{5 \cdot 6}
\]

\[
= \frac{5}{2}
\]

which is what we expected.

* We chose to multiply both sides by $\frac{3}{2}$ because that is the reciprocal of the coefficient of $\frac{2}{3}$, and their product is 1.

\[\square\]

Example 5.1.7. Solve $15 = -12 + 3x$ for $x$.

Solution: This time we will begin by isolating the $3x$ on the right-hand side.
\[ 15 = -12 + 3x \quad \text{Original equation} \]
\[ 12 + 15 = 12 - 12 + 3x \quad \text{Add 12 to both sides} \]
\[ 27 = 0 + 3x \quad \text{Simplify} \]
\[ 27 = 3x \quad \text{Simplify} \]
\[ \frac{1}{3} \cdot 27 = \frac{1}{3} \cdot 3x \quad \text{Multiply both sides by } \frac{1}{3} \]
\[ 9 = 1 \cdot x \quad \text{Simplify} \]
\[ 9 = x \quad \text{Simplify} \]

Therefore, our solution is \( x = 9 \). (Remember, \( 9 = x \) means the same thing as \( x = 9 \) because equality is symmetric.)

\[
\text{Example 5.1.8. Solve } \quad \frac{3x - 1}{y + 2} = 4 \]

for \( y \).

**Solution:** This one starts out with the variable we want to solve for in the denominator, so we will first multiply both sides by \( y + 2 \).

\[
\frac{3x - 1}{y + 2} \cdot (y + 2) = 4(y + 2)
\]
\[
3x - 1 = 4y + 8 \quad \text{Simplification and the distributive law}
\]
\[
3x - 1 - 8 = 4y + 8 - 8 \quad \text{Add } -8 \text{ to both sides}
\]
\[
3x - 9 = 4y \quad \text{Simplification}
\]
\[
\frac{1}{4} (3x - 9) = \frac{1}{4} (4y) \quad \text{Multiply both sides by } \frac{1}{4}
\]
\[
\frac{3}{4} x - \frac{9}{4} = 1y \quad \text{Simplification and the distributive law}
\]

Thus, \( y = \frac{3}{4} x - \frac{9}{4} \).

Ultimately, you will not want to have to write every step. When you have worked enough problems, you will see the patterns that emerge; certain things always happen. For example, to solve an equation like \( x + 6 = 8 \), we have been writing several steps.
\begin{align*}
x + 6 &= 8 \\
x + 6 - 6 &= 8 - 6 \\
x + 0 &= 8 - 6 \\
x &= 8 - 6.
\end{align*}

We have not simplified the right-hand side because we want to make a point. One who is comfortable with solving linear equations can go immediately from

\[ x + 6 = 8 \to x = 8 - 6 \]

without all of the intermediate steps. This is the level of comfort you are working toward. The more exercises you work, the sooner you will reach this stage!

We conclude this section with a brief discussion of quadratic equations.

**Theorem 31 (Zero-Product Theorem).** *If the product of two numbers is 0, then one of the numbers is 0. That is, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \).*

**Example 5.1.9.** This theorem allows us to solve an equation like \( x^2 + x - 6 = 0 \). We first draw on Chapter 4 to factor \( x^2 + x - 6 \). Factors of \(-6\) whose sum is 1 are 3 and \(-2\). Thus

\[ x^2 + x - 6 = (x - 2)(x + 3). \]

We now have two factors whose product is zero, so by the Zero-Product Theorem, one of the factors must be zero. That is, either \( x - 2 = 0 \) or \( x + 3 = 0 \). Both of these are linear equations, so we can solve them using the techniques of this section. We get that \( x = 2 \) or \( x = -3 \).

\[ \square \]

We do not want to go too far into solving equations like the one above, but we do want you to be aware that we now have the tools to attack such an equation, at least when we can factor. The end result is that we just have to solve a couple of linear equations.

**Example 5.1.10.** Solve \( 2x^2 + x - 3 = 0 \) for \( x \).

**Solution:** Since \( 2(-3) = -6 \), we need factors of \(-6\) whose sum is 1; these are 3 and \(-2\). Thus

\[ 2x^2 + x - 3 = 2x^2 - 2x + 3x - 3 = 2x(x - 1) + 3(x - 1) = (2x + 3)(x - 1). \]

We may now solve \((2x + 3)(x - 1) = 0\).

By the Zero-Product Theorem, we must have either \( 2x + 3 = 0 \) or \( x - 1 = 0 \). Therefore, either \( x = -\frac{3}{2} \) or \( x = 1 \).

\[ \square \]

**Section 5.1 Exercises**

Solve each linear equation.
1. \( x + 3 = 0 \)  
2. \( x + 15 = 0 \)  
3. \( x - 11 = 0 \)  
4. \( x - 12 = 7 \)  
5. \( x + 4 = 8 \)  
6. \( x - 3 = 8 \)  
7. \( x + 9 = -3 \)  
8. \( x - \frac{2}{3} = \frac{5}{3} \)  
9. \( x + 2\sqrt{5} = 7\sqrt{5} \)  
10. \( x + \frac{4}{5} = \frac{3}{7} \)  
11. \( x + 15 = -\frac{4}{3} \)  

12. \( x - \frac{9}{10} = \frac{2}{5} \)  
13. \( 2x + 6 = 0 \)  
14. \( 3x - 8 = 0 \)  
15. \( -5x + 12 = 0 \)  
16. \( 4x - 7 = 5 \)  
17. \( 12x + 8 = -4 \)  
18. \( 3x + 5 = 11 \)  
19. \( -2x + 4 = 7 \)  
20. \( 8x + \frac{6}{7} = \frac{2}{7} \)  
21. \( \frac{3}{8}x - 2 = 9 \)  
22. \( \frac{2}{3}x + \frac{4}{3} = -\frac{5}{3} \)  
23. \( 8x + 7 = 3x - 1 \)  
24. \( 4x + 2 = 11x + 5 \)  
25. \( 4 = 12 - x \)  
26. \( 3x + 5 = 7x - 2 \)  
27. \( -x + 8 = 4x - 3 \)  
28. \( \frac{4}{5}x - \frac{1}{4} = -\frac{3}{8}x + \frac{7}{4} \)  
29. \( 2x + \frac{7}{12} = \frac{5}{12} - \frac{1}{3}x \)  
30. \( 4x - 6 = \frac{7}{2}x + 1 \)

31. Paperback books cost 50 cents each at Peter’s bookstore. If Theresa paid $9.50 for paperbacks, how many did she buy?

32. If James had 6 more CD’s, he would have three times as many as Ellen. If Ellen has 7 CD’s, how many does James have?

33. Kathy’s stock portfolio has lost 45% of its value. If it is now worth $28,500, how much was it originally worth?

34. Bananas were on sale for 33 cents per pound. If Kim bought a frozen pizza for $3.95 and her total was 6.26, how many pounds of bananas did Kim buy?

Solve each equation for \( x \).

35. \( x^2 - 5x + 6 = 0 \)  
36. \( x^2 - 4x + 4 = 0 \)  
37. \( x^2 + 2x - 63 = 0 \)  
38. \( x^2 - 9 = 0 \)  
39. \( 6x^2 - x - 1 \)  
40. \( 8x^2 - 6x - 9 \)

### 5.2 Linear inequalities

We learn early on to compare the sizes of different quantities. If Timmy has 4 cookies and Johny only has 3, Johny is upset (and rightly so!). If the running back gains more than ten yards, the offense is happy and the defense is unhappy. If Phone Company A charges $15 per month and Phone Company B charges $18 per month for exactly the same services, we choose Phone Company A.
Sometimes, though, it is hard to make a direct comparison. Perhaps Phone Company A only charges $15 per month, but they also have a charge of 3 cents per minute of phone time. At what point does Phone Company B become cheaper?

In this section we will approach and answer such questions symbolically. Most of the techniques we found for solving linear equations will apply to solving these linear inequalities, as well.

**Definition 5.2.1.** Let \( a \) and \( b \) be real numbers. If \( b \) is larger than \( a \), then we write \( b > a \) (pronounced “\( b \) is greater than \( a \)” or \( a < b \) (pronounced “\( a \) is less than \( b \)”)

The symbols \( \geq \) (pronounced “greater than or equal to”) and \( \leq \) (pronounced “less than or equal to”) are used to indicate that the two numbers could be equal.

**Example 5.2.1.** \( 5 < 9, \ -7 < 2, \ 8 \leq 8, \ -\frac{2}{3} \geq -1 \).

We can see these inequalities on a number line, too. If \( a \) is to the right of \( b \) on a number line, then \( a > b \); if \( a \) is to the left of \( b \), then \( a < b \).

Suppose that we have two real numbers \( a \) and \( b \) such that \( a < b \). If we increase both \( a \) and \( b \) by the same amount, the results have the same relationship. For example, we would have \( a + 2 < b + 2 \) and \( a - 37 < b - 37 \). In general, we have the following theorem.

**Theorem 32.** Let \( a, b, \) and \( c \) be real numbers.

1. If \( a < b \), then \( a + c < b + c \).
2. If \( a > b \), then \( a + c > b + c \).

We can use this principle to solve linear inequalities.

**Example 5.2.2.** Solve \( x + 3 < 7 \) for \( x \).

**Solution:** We will add \(-3\) to both sides of the inequality.

\[
x + 3 < 7
\]
\[
x + 3 + (-3) < 7 + (-3)
\]
\[
x + 0 < 4
\]
\[
x < 4.
\]

Thus, our solution is \( \{ x | x < 4 \} \). This is shown on the number line below. The open circle around the 4 indicates that 4 is not part of the solution; the heavy line indicates the other values that solve the inequality.
Example 5.2.3. Solve $x + 4 \geq 2$ for $x$.

Solution: We will add $-4$ to both sides of the inequality.

\[
x + 4 \geq 2 \\
x + 4 + (-4) \geq 2 + (-4) \\
x + 0 \geq -2 \\
x \geq -2.
\]

That is, the values of $x$ that make $x + 4 \geq 2$ are those values of $x$ that are at least $-2$. We may write the solution as $\{x \mid x \geq -2\}$. The number line below shows the graph of this. This time, there is a darkened circle around $-2$, indicating that $-2$ is part of the solution.

Example 5.2.4. Let’s consider how to solve an inequality like

\[4x - 5 \geq 7.\]

From our experience with solving linear equations, we know that isolating the $4x$ is probably a good first step, so we will do that by adding 5 to both sides of the inequality.

\[
4x - 5 + 5 \geq 7 + 5 \\
4x \geq 12.
\]

If this were an equation instead of an inequality, we would multiply both sides by $\frac{1}{4}$ to solve for $x$. We would still like to do this, but we must first ask ourselves what the effect of multiplying both sides of the inequality by $\frac{1}{4}$ would be.
If \( a \) is less than \( b \), then one-fourth of \( a \) is less than one-fourth of \( b \). (Picture \( a \) as a smaller pie than \( b \), and take a quarter of both. Which quarter is bigger?) This means that we may multiply both sides of the inequality by \( \frac{1}{4} \).

\[
\frac{1}{4}(4x) \geq \frac{1}{4}(12) \\
x \geq 3.
\]

Thus, our solution is \( \{x | x \geq 3\} \), shown below. The darkened circle on 3 indicates that 3 is part of the solution.

![Figure 5.4: \( \{x | x \geq 3\} \)](image)

\[\square\]

The same principle we used for \( \frac{1}{4} \) applies to any positive factor. If \( a < b \), and if \( c \) is any positive number, then \( ac < bc \). Think of the factor of \( c \) as stretching or shrinking a line segment of length \( a \) and one of length \( b \). The one that was bigger is still bigger.

![Figure 5.5: Comparing \( ac \) and \( bc \)](image)

The lengths of the line segments are in proportion; the constant of proportionality is \( c \). Thus

\[
\frac{bc}{ac} = \frac{b}{a}.
\]

so if \( \frac{b}{a} > 1 \) (which is to say, \( b > a \)), then \( \frac{bc}{ac} \) is also greater than one (which is to say, \( bc > ac \)). This is summarized in the theorem below.

**Theorem 33.** Let \( a \) and \( b \) be real numbers, and let \( c \) be a positive real number.

1. If \( a < b \), then \( ac < bc \).

2. If \( a > b \), then \( ac > bc \).
Example 5.2.5. Solve \(3x + 8 > 2\) for \(x\).

**Solution:**

\[
\begin{align*}
3x + 8 & > 2 \\
3x + 8 - 8 & > 2 - 8 \quad \text{Subtracting 8 from both sides} \\
3x + 0 & > -6 \\
3x & > -6 \\
\frac{1}{3}(3x) & > \frac{1}{3}(-6) \quad \text{Multiplying both sides by} \ \frac{1}{3} \\
1 \cdot x & > -2 \\
x & > -2.
\end{align*}
\]

Thus our solution is \(\{x|x > -2\}\), shown below. The open circle around \(-2\) indicates that \(-2\) is *not* part of the solution.

![Figure 5.6: \(\{x|x > -2\}\)](image)

Example 5.2.6. Solve \(\frac{2}{3}x + \frac{1}{4} \leq \frac{1}{3}x - \frac{4}{3}\).
Solution:

\[
\frac{3}{5}x + \frac{1}{4} \leq \frac{1}{3}x - \frac{4}{3}
\]

Subtracting \(\frac{1}{4}\) from both sides

\[
\frac{3}{5}x + \frac{1}{4} - \frac{1}{4} \leq \frac{1}{3}x - \frac{4}{3} - \frac{1}{4}
\]

Finding a common denominator

\[
\frac{3}{5}x + 0 \leq \frac{1}{3}x - \frac{16}{12} - \frac{3}{12}
\]

\[
\frac{3}{5}x \leq \frac{1}{3}x - \frac{19}{12}
\]

Adding \(-\frac{1}{3}x\) to both sides

\[
\frac{-1}{3}x + \frac{3}{5}x \leq -\frac{1}{3}x + \frac{1}{3}x - \frac{19}{12}
\]

Combining like terms

\[
\frac{4}{15}x \leq -\frac{19}{12}
\]

Adding fractions

\[
\frac{15}{4} \cdot \frac{4}{15}x \leq \frac{15}{4} \left( -\frac{19}{12} \right)
\]

Multiplying both sides by \(\frac{15}{4}\)

\[
1 \cdot x \leq \frac{-3 \cdot 5 \cdot 19}{24 \cdot 3}
\]

Multiplying fractions and factoring

\[
x \leq -\frac{57}{16}
\]

Simplifying fractions.

Thus, our solution is \(\{x | x \leq -\frac{57}{16}\}\). The closed circle around \(-\frac{57}{16}\) indicates that \(-\frac{57}{16}\) is part of the solution.

Figure 5.7: \(\{x | x \leq -\frac{57}{16}\}\)

When we have a negative factor, we run into difficulties.

Example 5.2.7. We know that \(4 < 5\). However, \(4(-3) = -12\) and \(5(-3) = -15\), so \(4(-3) > 5(-3)\). We have reversed the direction of the inequality! What is happening?

Remember that multiplying by a negative number changes the sign of the number you had. If one number is farther to the right than another on the number line, changing both of their signs moves that one farther to the left. In the figure below, we have \(a < b\), but \(-a > -b\).
That this will always happen is recorded in the next theorem.

**Theorem 34** (Multiplication by a Negative). Let \(a\) and \(b\) be real numbers, and let \(c\) be a negative real number.

1. If \(a < b\), then \(ac > bc\).
2. If \(a > b\), then \(ac < bc\).

**Example 5.2.8.** Solve \(-4x < -8\) for \(x\).

**Solution:** We will need to multiply both sides of the inequality by \(-\frac{1}{4}\), which will reverse the sense of the inequality.

\[
\begin{align*}
-4x &< -8 \\
-\frac{1}{4}(-4x) &> -\frac{1}{4}(-8) & \text{Multiplying both sides by } -\frac{1}{4} \text{ \textbf{NOTE CHANGE}} \\
1 \cdot x &> 2,
\end{align*}
\]

so \(\{x|x > 2\}\) is our solution. Notice that the direction of the inequality changed, as the Multiplication by a Negative Theorem says it must.

We interrupt this lesson for a harangue.

**CAUTION:** Some students are tempted to skip some steps in this by writing something like

\[
\begin{align*}
-4x &< -8 \\
\overline{-4} & \overline{<} \overline{-4}.
\end{align*}
\]

**Don’t do this!** It’s sloppy, lazy, and wrong! What’s wrong with it, you ask? We didn’t change the direction of the inequality! You might or might not remember to correct that on the next line, but *this* line is incorrect. Instead, take the time to write the one extra line to
make it correct. It will save you from making errors and it will make your solutions much easier to follow.

Many students also try something like that when solving an equation (when division by a negative doesn’t change anything) or when dividing by a positive (which leaves an inequality alone). **Don’t do it then, either!** It may not be wrong, but it is still sloppy and lazy, and it will encourage bad habits. The “extra” line it takes to do it carefully is well worth the effort.

We now return you to your regularly scheduled lesson.

**Example 5.2.9.** Solve $2x + 7 < 5x + 3$.

**Solution:**

\[
2x + 7 < 5x + 3 \\
2x + 7 - 7 < 5x + 3 - 7 \quad \text{Subtracting 7 from both sides} \\
2x < 5x - 4 \quad \text{Simplifying} \\
-5x + 2x < -5x + 5x - 4 \quad \text{Adding$-5x$ to both sides} \\
-3x < -4 \quad \text{Combining like terms} \\
-\frac{1}{3}(-3x) > -\frac{1}{3}(-4) \quad \text{Multiplying both sides by$-\frac{1}{3}$ NOTE CHANGE} \\
x > \frac{4}{3}.
\]

Again, multiplying both sides by a negative number changed the direction of the inequality. Our solution is $\{x|x > \frac{4}{3}\}$, shown below.

\[\begin{array}{c}
\text{0} \\
\text{4}/\text{3} \\
\text{-1} \\
\text{-2} \\
\text{-3}
\end{array}\]

Figure 5.10: $\{x|x > \frac{4}{3}\}$

**Example 5.2.10.** Let’s conclude this section by solving the problem set out at the beginning. We restate it here for convenience.

Suppose Phone Company A only charges $15 per month, but they also have a charge of 3 cents per minute of phone time. Phone company B charges a flat rate of $18 per month. At what point does Phone Company B become cheaper?

**Solution:** Let’s let $t$ represent the number of minutes of phone time a customer expects to use in a month. Then Company A will charge $15 + 0.03t$ dollars for the month, while Company B still charges 18 dollars. We want to solve

\[15 + 0.03t > 18\]

for $t$. (That says that Company A’s charge is greater than Company B’s.)
\[
15 + 0.03t > 18 \\
-15 + 15 + 0.03t > -15 + 18 \\
0.03t > 3 \\
\frac{1}{0.03}(0.03t) > \frac{1}{0.03}(3) \\
t > 100.
\]

Thus, after 100 minutes (1 hour and 40 minutes), Company B becomes cheaper.

\section*{Section 5.2 Exercises}

Graph each set on a number line.

1. \{\(x \mid x > 1\}\} 
2. \{\(x \mid x \leq -1\}\} 
3. \{\(x \mid x \geq 4\}\} 
4. \{\(x \mid x < \frac{5}{2}\}\} 
5. \{\(x \mid x > -3\}\} 
6. \{\(x \mid x \geq 1.5\}\}

Express in set-builder notation the region graphed.

7.
8.
9.
10.
11.
12.

Solve each linear inequality and graph your solution.

13. \(x + 6 > 5\) 
14. \(x - 3 \geq 2\) 
15. \(x + 4 \leq -1\) 
16. \(x - \frac{1}{2} < 4\) 
17. \(3x > 7\) 
18. \(3x + 4 < -2\) 
19. \(5x - 2 \geq 9\) 
20. \(2x + 1 > 3\) 
21. \(-2x + 4 < 7\)
22. \(-5x - 1 \leq 2\)  
25. \(2x + 5 > 9x - 3\)  
28. \(\frac{3}{8}x + 1 < \frac{2}{5}x + \frac{5}{3}\) 
23. \(\frac{4}{5}x + 2 \leq -3\)  
26. \(x + 4 < 7x\)  
29. \(-3x + 5 \leq x + 8\) 
24. \(-\frac{1}{4}x + \frac{1}{2} < \frac{3}{8}\)  
27. \(2x - 2 \geq 4x + 12\)  
30. \(14x + \frac{11}{8} < \frac{16}{3}x - 2\)

31. A car rental company offers two types of rentals. The first type costs $25 per day plus 40 cents per mile. The second type costs $60 per day for unlimited mileage. After how many miles is the second type cheaper?

32. A tool rental company charges 20 dollars per hour for a front-end loader. A competitor offers the same front-end loader for 95 dollars plus 10 dollars per hour. After how many hours is the second company cheaper?

33. One long-distance plan costs $18 per month plus 7 cents per minute. A second plan costs $25 per month plus 5 cents per minute. After how many minutes is the second plan cheaper?

34. Membership in a book club costs $40 per year, and members receive a 10% discount on books. A competing book club charges a membership of 12$ per year, but members receive only a 5% discount. How much must one spend before the first book club is the better deal? [Hint: with a 10% discount, a book normally costing \(x\) dollars will cost 0.9\(x\) dollars.]

5.3 The Rectangular Coordinate System

In this section, we offer an introduction to the rectangular coordinate system, also known as the Cartesian coordinate system after the French mathematician and philosopher René Descartes. Nowadays, it seems like a fairly simple idea, but is an idea that revolutionized mathematics and dramatically changed the world. The invention of the Cartesian coordinate system made it possible to discover analytic geometry and then the Calculus. Both have played a major role in creating today’s world.

The story goes that as Descartes was recuperating in an army hospital, he observed a fly buzzing about the room. It suddenly struck him that he could describe the position of the fly in terms of its distances from the floor and two adjacent walls. It is this idea (in only two dimensions) that we explore in this section.

Early in elementary school, we learned to use a number line to represent real numbers. Consider any straight line. It is customary to draw this line horizontally on the page.

![Figure 5.11: A Horizontal Line](image)

To each point on the line, we associate a unique real number as follows:
1. Choose a point $O$ on the line, and label it with a 0 (zero). We will call $O$ the **origin**.

![Figure 5.12: Label the origin](image)

2. Now choose any other point and label it with a 1. The distance from this second point to the origin establishes the size of one unit. Customarily, we choose this point to lie to the right of the origin.

![Figure 5.13: Label the point 1](image)

With these markings, we can lay out all of the integers on the line by marking divisions of this size. Note that the negatives are on one side of 0, and the positives are on the other side of 0.

![Figure 5.14: The Number Line](image)

To obtain rational numbers, just divide the unit segments into appropriate lengths, as shown.

The irrational numbers are harder; we will not deal with the construction of the rest of the real numbers, but assume that every real number corresponds to exactly one point on the line, and every point on the line corresponds to exactly one real number. The number corresponding to a given point is its **coordinate**.

The distance between two points on the line is the (positive) difference in their coordinates.

*Example 5.3.1.* The distance from 7 to 4 is 3 units, and so is the distance from 4 to 7.

*Example 5.3.2.* The distance from $-2$ to 5 is $5 - (-2) = 7$ units.
To describe points in a plane, rather than just on a line, we require two coordinates. We begin with two number lines; for convenience, we will refer to one line as the \( x \)-axis and the other as the \( y \)-axis. It is common, though not necessary, to choose the coordinates on the two lines so that the units are equal.

![Figure 5.15: The \( x \)- and \( y \)-axes](image)

We then place these two lines in a plane so that they intersect (a) at right angles and (b) at the point \( O \), which we will still refer to as the origin. Traditionally, we place the \( x \)-axis horizontally on the page with positive numbers to the right, and the \( y \)-axis is placed vertically on the page with positive numbers going up. For simplicity, we label the axes with just \( x \) and \( y \).

![Figure 5.16: The \( x \)- and \( y \)-axes in position](image)

This establishes the coordinate system; notice that the lines divide the plane of the page into four sections, called quadrants. These are numbered with roman numerals I, II, III, and IV counterclockwise, as indicated.

Let us now see how we can use the coordinate system to identify points in the plane. Imagine that through each point on the \( y \)-axis there is a horizontal line (parallel to the \( x \)-axis). Likewise, imagine that through each point on the \( x \)-axis, there is a vertical line.
(parallel to the $y$-axis). Together, these lines form a grid. In the figure below, we have only shown those lines passing through integer points on each axis, but every point on the $y$-axis has a horizontal line through it (even 1.3246 and $-\pi$), and every point on the $x$-axis has a vertical line through it.

![Figure 5.17: Grid formed by lines parallel to the axes](image)

These grid lines fill the entire plane, so no matter what point in the plane we choose, it will lie on the intersection of two of these lines. Furthermore, it will lie on only one of these points of intersection. This means that we can identify the point with numbers the two lines pass through, as in the figure below.

*Example 5.3.3.* The point $P$ is shown in the figure.

![Figure 5.18: The point $(2, 4)$](image)

Since $P$ lies on the intersection of the vertical line through 2 and the horizontal line through 4, we will label $P$ with the *ordered pair* $(2, 4)$. The 2 is the *$x$-coordinate* and the
4 is the \textit{y-coordinate}. Collectively, 2 and 4 are known as the \textit{coordinates} of the point \( P \). It is customary to write the \( x \)-coordinate first.

In summary, we label the point \( P \) with the coordinates \((2, 4)\) because to reach \( P \) from the origin, we go to the right 2 units and up 4 units.

\[ \square \]

\textit{Example} 5.3.4. On the other hand, if we are given coordinates like, say, \((4, -1)\), how do we find the corresponding point? Based on the way we built the coordinate system, the point \((4, -1)\) will lie on the vertical line passing through the 4 on the \( x \)-axis and the horizontal line passing through the \(-1\) on the \( y \)-axis. Thus, we count left 4 units from the origin and down 1 unit from there.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.19}
\caption{The point \((4, -1)\)}
\end{figure}

There are two main points to keep in mind. First, the first of the two coordinates gives the horizontal distance from the origin and the second gives the vertical distance from the origin. Second, positive coordinates go up or to the right, and negative coordinates go down or to the left. Other than that, it’s just counting units.

\textit{Example} 5.3.5. Locate the points \((-1, 3), (2, 5), (-1, -1), (4, -2)\) and \( \left(\frac{3}{4}, -\frac{1}{2}\right) \) in the plane.
Figure 5.20: Points with coordinates

Notice that \((-1, 3)\) and \((-1, -1)\) have the same \(x\)-coordinate and that they line on a vertical line (parallel to the vertical axis). Also, this is the first time in this text that we have used the notation \(a/b\) for the fraction \(\frac{a}{b}\). This is a common notation, but it can lead to ambiguity if it is not used carefully, so we use it as little as possible.

We have omitted a small but important detail up to now: what do we do with points on the axes? This is really no problem; we simply label them with one coordinate of 0.

\textit{Example 5.3.6}. The figure shows the points \((3, 0)\) and \((0, -2)\). For \((3, 0)\), we must simply go to the right 3 units and up 0 units, or not at all. This means that we stay on the \(x\)-axis. For \((0, -2)\), we go to the right 0 units (or not at all) and down 2 units. (Up \(-2\) units and down 2 units are the same thing.)
Section 5.3 Exercises

1. Determine as closely as possible the coordinates of the points plotted.

Plot each point.

17. (1,3)  21. (6,6)  25. (−1.5, 1.75)  29. \( \left( \frac{1}{7}, \frac{2}{7} \right) \)
18. (4, −1)  22. (−2, −2)  26. (0, 8)  30. (4, 2)
19. (3, 2)  23. (−3.5, 1)  27. (0, −3)  31. (3.75, −4)
20. (0.5, −4)  24. (2.25, 0)  28. (1, 2)  32. (−5, −3.75)

5.4 The Slope of a Line

Suppose that we have a line in the plane and that we have lain down our axes.
If we lay down horizontal and vertical lines as shown below, we see that we create triangles that are in proportion.

This means that the ratio of the sides of the small triangle and the ratio of the sides of the large triangle are equal:

\[
\frac{1}{2} = \frac{2}{4},
\]

which we know is true.

The remarkable fact is that the ratio of the sides will be \( \frac{1}{2} \) no matter where along the line we place our triangle! This means that that quantity reflects something special about the line. Since it is special, we give it a name. The ratio is called the \textit{slope} of the line. Thus, the slope of the line shown above is \( \frac{1}{2} \).
Definition 5.4.1. Suppose that \((x_1, y_1)\) and \((x_2, y_2)\) are two different points on a line. The slope of the line is the ratio

\[
m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

Examination of the triangles shows that we can use any two points we want; the slope will come out the same. Many textbooks refer to the change in the \(y\)-values, \(y_2 - y_1\), as the rise since that is how far “up” (the page) the line goes for a certain horizontal change. Likewise, the change in the \(x\)-values, \(x_2 - x_1\), is the run; it refers to how far horizontally the line moves. With this terminology, the slope may be thought of as the “rise over the run.”

![Figure 5.24: The slope of a line is the rise over the run](image)

Example 5.4.1. Find the slope of the line shown.

![Figure 5.25: Find the slope of the line](image)
To determine the rise and the run, we need to locate two points on the line, and any two points will do. Our computations will be easier if we can find points with integer coordinates. It appears that the points \((-1, 3)\) and \((2, -3)\) are on the line, so the rise is \(-3 - 3 = -6\) and the run is \(2 - (-1) = 3\). Therefore, the slope of the line is \(-\frac{6}{3} = -2\).

\[\square\]

**Example 5.4.2.** Find the slope of the line through the points \((4, 1)\) and \((-2, -3)\).

We may apply our slope formula:

\[
\frac{1 - (-3)}{4 - (-2)} = \frac{4}{6} = \frac{2}{3}.
\]

Thus, the slope of this line is \(\frac{2}{3}\).

“But wait!” you say. “I thought \(x_2\) and \(y_2\) came first in the formula!” True, but how can you tell which is which? In fact, it doesn’t matter. In the first place, the line through the points \((4, 1)\) and \((-2, -3)\) should (and does) have the same slope as the line through \((-2, -3)\) and \((4, 1)\), so it doesn’t matter which you think of as the “first” point and which you think of as the “second” point.

Also, algebraically,

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2},
\]

which also says that either point can come first. In our example, we could have computed

\[
\frac{-3 - 1}{-2 - 4} = \frac{-4}{-6} = \frac{2}{3},
\]

and we again find a slope of \(\frac{2}{3}\).

**Note:** It doesn’t matter which point you think of as the first point, but once you decide on one, you need to stick with it.

\[\square\]

**Example 5.4.3.** Find the slope of the line through the points \((2.5, -3.3)\) and \((-1.6, -3.2)\).

\[
\frac{-3.3 - (-3.2)}{2.5 - (-1.6)} = \frac{-0.1}{4.1}.
\]

\[\square\]

**Example 5.4.4.** Find the slope of the line through the points \((3, 2)\) and \((-4, 2)\).

\[
\frac{2 - 2}{3 - (-4)} = \frac{0}{7} = 0,
\]

so the slope is 0. The line through these points is horizontal.
If the slope of a line is \( \frac{4}{3} \), then for every 3 steps to the right, the line rises 4 units. Likewise, if the slope of the line is \( -\frac{4}{3} \), then for every 3 steps to the right, the line falls 4 units. That is, when the slope of the line is positive, it rises to the right; when the slope is negative, it falls to the right. In between positive and negative slopes is zero, and a line with slope zero neither rises nor falls to the right.

In addition, a line with a slope of \( \frac{7}{3} \) goes up 7 units for every 3 units it moves to the right; a line with a slope of \( \frac{4}{3} \) only goes up 4 units for every 3 units to the right. That is, for positive slopes, the line with the greater slope is steeper (and rises faster).

On the other hand, a line with a slope of \( -\frac{7}{3} \) goes down 7 units for every 3 units it moves to the right, and a line with a slope of \( -\frac{4}{3} \) goes down 5 units for every 3 units to the right.
That is, for negative slopes, the line with the smaller slope is steeper (and falls faster).

Figure 5.28: The effects of the slope on the behavior of a line

Example 5.4.5. Determine whether the three points \((8, 1), (4, 6),\) and \((0, 10)\) are collinear. (That is, do they lie on the same line?)

Solution: If they lie on the same line, then the slope from \((0, 10)\) to \((4, 6)\) will be the same as the slope from \((4, 6)\) to \((8, 1)\) since the slope of the line is the same no matter which two points on the line we choose.

1. From \((0, 10)\) to \((4, 6)\): \(\frac{10 - 6}{0 - 4} = \frac{4}{-4} = -1.\)

2. From \((4, 6)\) to \((8, 1)\): \(\frac{6 - 1}{4 - 8} = \frac{5}{-4} = -\frac{5}{4}.\)

Since these slopes are different, the points do not lie on the same line.
Visually, it is close; that’s why it’s nice to have an analytic description of slope. We can’t always trust our eyes, especially when the points are far apart.

We know from our study of fractions that if the denominator is zero, we have a problem. Can this occur? That is, is it possible that $x_1$ and $x_2$ are equal? If they are, then the denominator, $x_1 - x_2$, will be zero and the slope will be undefined. Yes, it can happen: if a line is vertical, then the $x$-values of all points on the line are the same. In this case, we say that the line has no slope.

**NOTE:** There is a vast difference between having no slope and having a slope of 0. We saw above that a line with a slope of 0 is horizontal. Now we see that a line with no slope is vertical. Understanding the slope formula will help you keep these ideas straight.
If a line is not vertical, then no two points on the line will have the same $x$-value, and the slope will be defined. This gives us the following theorem.

**Theorem 35.** The slope of a line is undefined if and only if the line is vertical.

We have discussed above what the slope of a line can tell you about the line. What can comparing the slopes of two lines tell you?

**Theorem 36.** Two lines are parallel if and only if they have equal slopes.

This is a reasonable claim: if two lines have equal slopes, then every step to the right will cause both lines to rise exactly the same amount, so they can never intersect. On the other hand, if the two lines have different slopes, then one will rise faster than the other and they will have to meet somewhere, so they can’t be parallel.

**Example 5.4.6.** Determine whether a line with a slope of $\frac{-3}{4}$ is parallel to the line shown.

![Figure 5.31: Does this line have slope $\frac{-3}{4}$?](image)

The line appears to pass through the points $(-2, 4)$ and $(2, 1)$; from this, we can determine its slope.

$$\frac{4 - 1}{-2 - 2} = \frac{3}{-4} = \frac{-3}{4}.$$  

Thus, any other line with slope $\frac{-3}{4}$ will be parallel to this one.

There is also a very attractive relationship between the slopes of perpendicular lines.

**Theorem 37.** Two lines, neither of which is vertical, are perpendicular if and only if the product of their slopes is $-1$.

This is a more sophisticated theorem; we will simply accept its truth for this course. You may see why this is true if you take a course in analytic geometry.
Example 5.4.7. Are the lines shown perpendicular?

![Diagram of two lines with points (2,2) and (4,3) and (2,2) and (1,4)]

Figure 5.32: Are the lines perpendicular?

The lines do look perpendicular, but that isn’t good enough. Perhaps the angle between them is 89.999 degrees! We will use the theorem to find out for sure.

The line that rises to the right (and thus has positive slope) passes through the points (2, 2) and (4, 3), so its slope is

\[ \frac{2 - 3}{4 - 6} = \frac{-1}{-2} = \frac{1}{2}. \]

The line that falls to the right (and thus has negative slope) passes through the points (2, 2) and (1, 4), so its slope is

\[ \frac{2 - 4}{2 - 1} = \frac{-2}{1} = -2. \]

Since the product of these is

\[ (-2) \cdot \frac{1}{2} = -1, \]

the two lines are perpendicular.

\[ \square \]

Section 5.4 Exercises

Determine whether the slope \( m \) of each line is positive, negative, zero, or undefined \textit{without} actually computing the slope of the line.
Determine the slope $m$ of each line shown.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16.
On a set of axes, draw two lines with the given slope $m$. One line should pass through the given point $P$, and the other should not.

10. $m = 3; P = (2, 1)$  
12. $m = -\frac{2}{3}; P = (1, 4)$  
14. $m = 1; P = (0, 0)$  
11. $m = \frac{1}{3}; P = (-5, 2)$  
13. $m = \frac{5}{3}; P = (-2, -1)$  
15. $m = -6; P = (-5, 8)$

Determine the slope $m$ of the line through the given points. If the slope is undefined, say so.

16. $(2, 1)$ and $(3, -2)$  
20. $(2, 7)$ and $(2, -1)$  
24. $(\frac{1}{4}, \frac{2}{7})$ and $(\frac{7}{3}, \frac{3}{7})$  
17. $(4, 6)$ and $(1, 0)$  
21. $(0, 0)$ and $(5, 3)$  
25. $(1, -12)$ and $(3, 6)$  
18. $(5, 2)$ and $(-3, 2)$  
22. $(0, 0)$ and $(-2, -7)$  
26. $(4, -2)$ and $(4, 12)$  
19. $(\frac{3}{5}, \frac{4}{7})$ and $(\frac{8}{9}, \frac{9}{17})$  
23. $(\frac{5}{8}, \frac{2}{3})$ and $(\frac{4}{3}, \frac{1}{6})$  
27. $(3, 5)$ and $(8, -2)$

Find (a) the slope $a$ of a line parallel to a line with the given slope and (b) the slope $b$ of a line perpendicular to a line with the given slope $m$.

28. $m = 5$  
30. $m = -\frac{5}{2}$  
32. $m = -8$  
34. $m = -\frac{7}{4}$  
29. $m = \frac{2}{3}$  
31. $m = -1$  
33. $m = \frac{4}{15}$  
35. $m = 0$

### 5.5 Equations of Lines

In the last section, we observed that the slope of a line gives us a great deal of information about the line. However, it does not give us all the information about a line. Specifically, since parallel lines have the same slope, the slope only tells us that our line is one of a large collection of parallel lines. We need another piece of information in order to pin down just which line is ours.
Figure 5.33: Parallel lines

From the figure above, it is fairly clear that each of the parallel lines intersects the $y$-axis exactly once, and no two of them intersect the $y$-axis in the same point. (Otherwise, they would also intersect each other at that point, which parallel lines don’t do!) This means that that intersection point with the $y$-axis is one additional piece of information that will tell us exactly which line is ours.

**Definition 5.5.1.** The point of intersection of a non-vertical line with the $y$-axis is called the *y-intercept* of the line. It is the point on the line having an $x$-coordinate of 0.

We had to specify “non-vertical” because a vertical line either *is* the $y$-axis or is parallel to the $y$-axis; either way, “the” $y$-intercept doesn’t make sense.

*Example 5.5.1.* The $y$-intercept of the line shown is $(0,3)$, or just 3, since the $x$-coordinate is automatically 0 for the $y$-intercept.

Figure 5.34: The $y$-intercept is $(0,3)$
Now take a line with slope $m$, and suppose that $(x, y)$ is some generic point on that line. Also suppose that the $y$-intercept of the line is the point $(0, b)$. (Remember, the $y$-intercept is where the $x$-coordinate is 0.) Since we have two points on the line, we can compute the slope, which we already know is $m$:

$$\frac{y - b}{x - 0} = m,$$

or

$$\frac{y - b}{x} = m.$$

If we multiply both sides of this equation by $x$, we see that

$$\frac{y - b}{x} \cdot x = m \cdot x$$

$$y - b = mx$$

$$y = mx + b.$$

This is our first equation for a line.

**Theorem 38.** If $(x, y)$ is a point on the line with slope $m$ and $y$-intercept $(0, b)$, then $x$ and $y$ satisfy the equation

$$y = mx + b.$$

This equation is called the **slope-intercept** equation of the line.

It is called the slope-intercept equation of the line because we need to know the slope of the line and the $y$-intercept.

**Example 5.5.2.** Write an equation of the line with slope $-\frac{3}{4}$ and $y$-intercept 5.

**Solution:** We may simply plug into the slope-intercept form to find the equation

$$y = -\frac{3}{4}x + 5.$$

The graph of this line is shown below.

![Graph of the line $y = -\frac{3}{4}x + 5$](image)

Figure 5.35: The line $y = -\frac{3}{4}x + 5$
Example 5.5.3. The equation \( y = -\frac{2}{3}x + 2 \) is an equation of a line. Find the slope and the \( y \)-intercept of this line.

**Solution:** Since the line is in slope-intercept form, we may simply read off the slope and the \( y \)-intercept. The slope is \(-\frac{2}{3}\) and the \( y \)-intercept is 2.

Example 5.5.4. Find an equation of the line with slope 0 and \( y \)-intercept \(-3\).

**Solution:** We have \( y = (0)x - 3 = -3 \), or just \( y = -3 \).

The previous example shows that a horizontal line with \( y \)-intercept \( c \) has the form \( y = c \). That is, no matter what value of \( x \) we choose, the \( y \)-value of a point on the line is \( c \). Similarly, if a line is vertical (and therefore has no slope), then all of the \( x \)-coordinates of points on that line are equal, and the line has equation \( x = c \).

![Horizontal and vertical lines](image-url)

Figure 5.36: Horizontal and vertical lines

Example 5.5.5. Find an equation of the line through the point \((-2, 1)\) that has \( y \)-intercept \((0, 3)\).

**Solution:** This time, we are not given the slope, but we have two points that we can use to find the slope \( m \):

\[
m = \frac{1 - 3}{-2 - 0} = \frac{-2}{-2} = 1.
\]

Now we can go to our slope-intercept form and find the equation

\[
y = 1 \cdot x + 3,
\]

or

\[
y = x + 3.
\]
What if we are not given the \( y \)-intercept? For example, suppose we know that the slope of the line is \( \frac{7}{12} \) and that the line passes through the point \((3, 5)\). If \((x, y)\) is another (generic) point on the line, then the slope of the line is

\[
\frac{y - 5}{x - 3}.
\]

Since we already know that the slope is \( \frac{7}{12} \), these two must be equal:

\[
\frac{y - 5}{x - 3} = \frac{7}{12}.
\]

Let’s multiply both sides by \( x - 3 \) to simplify the left-hand side.

\[
\frac{y - 5}{x - 3} \cdot (x - 3) = \frac{7}{12} (x - 3)
\]

\[
y - 5 = \frac{7}{12} (x - 3).
\]

That is, if \((x, y)\) is another point on the line, then \( x \) and \( y \) make the equation

\[
y - 5 = \frac{7}{12} (x - 3)
\]

true. This is an example of an equation of a line in point-slope form. This is the form we typically use when we are given the slope of a line and a point on the line.

**Theorem 39.** If \((x, y)\) is a point on the line with slope \( m \) passing through the point \((x_1, y_1)\), then \( x \) and \( y \) satisfy the equation \( y - y_1 = m(x - x_1) \). This equation is called a point-slope equation of the line.

If \((x_1, y_1)\) is a given point, \( m \) is the given slope, and \((x, y)\) is any other point on the line, then we must have

\[
\frac{y - y_1}{x - x_1} = m
\]

by the definition of slope. If we multiply both sides of this by \( x - x_1 \), we get

\[
\frac{y - y_1}{x - x_1} (x - x_1) = m(x - x_1)
\]

\[
y - y_1 = m(x - x_1),
\]

which is the point-slope form above.

*Example 5.5.6.* Find an equation of the line with slope \(-5\) that passes through the point \((3, -2)\).

**Solution:** One such equation is

\[
y - (-2) = -5(x - 3)
\]

by our point-slope theorem. We can simplify this to

\[
y + 2 = -5(x - 3).
\]
Example 5.5.7. Find an equation of the line with slope $-\frac{5}{3}$ that passes through the point $(-2, -1)$.

Solution: One such equation is $y - (-1) = -\frac{5}{3}(x - (-2))$, or $y + 1 = -\frac{5}{3}(x + 2)$. We can also solve this for $y$, which is equivalent to rewriting it in slope-intercept form.

\[
y + 1 = -\frac{5}{3}(x + 2)
\]

\[
y + 1 = \frac{5}{3}x - \frac{10}{3} - 1
\]

\[
y = \frac{5}{3}x - \frac{13}{3}.
\]

Suppose now that we are simply given two points on the line. We have already seen that this will give us the slope, so we really have (in effect) the slope and a point (and another point). This should be enough information to find an equation of the line.

Example 5.5.8. Find an equation of the line through the points $(3, 5)$ and $(7, -1)$.

Solution: First, we need the slope:

\[
m = \frac{5 - (-1)}{3 - 7} = \frac{6}{-4} = -\frac{3}{2}.
\]

Now we need to choose a point to use our point-slope form. Let’s try them both and see what happens.

1. Using $(3, 5)$: $y - 5 = -\frac{3}{2}(x - 3)$.

2. Using $(7, -1)$: $y - (-1) = -\frac{3}{2}(x - 7)$, or $y + 1 = -\frac{3}{2}(x - 7)$.

Oh, dear. These don’t seem to be the same! How can we decide whether or not they really are? One way might be to solve both of them for $y$ (and thereby put them both in slope-intercept form) to see if we get the same thing.
\[
\begin{align*}
\ y - 5 &= -\frac{3}{2}(x - 3) \quad \text{First equation} \\
\ y - 5 &= -\frac{3}{2}x + \frac{3}{2}(3) \quad \text{Distributive law} \\
\ y - 5 &= -\frac{3}{2}x + \frac{9}{2} \quad \text{Simplification} \\
\ y - 5 + 5 &= -\frac{3}{2}x + \frac{9}{2} + 5 \quad \text{Adding 5 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{9}{2} + \frac{10}{2} \quad \text{Finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Addition of fractions}
\end{align*}
\]

Now let’s try it for the second equation.

\[
\begin{align*}
\ y + 1 &= -\frac{3}{2}(x - 7) \quad \text{Second equation} \\
\ y + 1 &= -\frac{3}{2}x + \frac{3}{2}(7) \quad \text{Distributive law} \\
\ y + 1 &= -\frac{3}{2}x + \frac{21}{2} \quad \text{Simplification} \\
\ y + 1 - 1 &= -\frac{3}{2}x + \frac{21}{2} - 1 \quad \text{Add -1 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{21}{2} - \frac{2}{2} \quad \text{Simplification and finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Subtraction of fractions}
\end{align*}
\]

How wonderful! Both slope-intercept equations are the same! This means that when we have two points on the line, we may choose either one to use in our point-slope equation.

\[
\begin{align*}
\ y - 5 &= \frac{3}{2}(x - 3) \quad \text{First equation} \\
\ y - 5 &= -\frac{3}{2}x + \frac{3}{2}(3) \quad \text{Distributive law} \\
\ y - 5 &= -\frac{3}{2}x + \frac{9}{2} \quad \text{Simplification} \\
\ y - 5 + 5 &= -\frac{3}{2}x + \frac{9}{2} + 5 \quad \text{Adding 5 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{9}{2} + \frac{10}{2} \quad \text{Finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Addition of fractions}
\end{align*}
\]

Now let’s try it for the second equation.

\[
\begin{align*}
\ y + 1 &= -\frac{3}{2}(x - 7) \quad \text{Second equation} \\
\ y + 1 &= -\frac{3}{2}x + \frac{3}{2}(7) \quad \text{Distributive law} \\
\ y + 1 &= -\frac{3}{2}x + \frac{21}{2} \quad \text{Simplification} \\
\ y + 1 - 1 &= -\frac{3}{2}x + \frac{21}{2} - 1 \quad \text{Add -1 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{21}{2} - \frac{2}{2} \quad \text{Simplification and finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Subtraction of fractions}
\end{align*}
\]

How wonderful! Both slope-intercept equations are the same! This means that when we have two points on the line, we may choose either one to use in our point-slope equation.

\[
\begin{align*}
\ y - 5 &= \frac{3}{2}(x - 3) \quad \text{First equation} \\
\ y - 5 &= -\frac{3}{2}x + \frac{3}{2}(3) \quad \text{Distributive law} \\
\ y - 5 &= -\frac{3}{2}x + \frac{9}{2} \quad \text{Simplification} \\
\ y - 5 + 5 &= -\frac{3}{2}x + \frac{9}{2} + 5 \quad \text{Adding 5 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{9}{2} + \frac{10}{2} \quad \text{Finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Addition of fractions}
\end{align*}
\]

Now let’s try it for the second equation.

\[
\begin{align*}
\ y + 1 &= -\frac{3}{2}(x - 7) \quad \text{Second equation} \\
\ y + 1 &= -\frac{3}{2}x + \frac{3}{2}(7) \quad \text{Distributive law} \\
\ y + 1 &= -\frac{3}{2}x + \frac{21}{2} \quad \text{Simplification} \\
\ y + 1 - 1 &= -\frac{3}{2}x + \frac{21}{2} - 1 \quad \text{Add -1 to both sides} \\
\ y &= -\frac{3}{2}x + \frac{21}{2} - \frac{2}{2} \quad \text{Simplification and finding a common denominator} \\
\ y &= -\frac{3}{2}x + \frac{19}{2} \quad \text{Subtraction of fractions}
\end{align*}
\]

How wonderful! Both slope-intercept equations are the same! This means that when we have two points on the line, we may choose either one to use in our point-slope equation.

This leads us to the two-point form of a line, which is really just a shortcut for what we did in Example 5.5.8.

**Theorem 40.** If \((x, y)\) is a point on the line through the points \((x_1, y_1)\) and \((x_2, y_2)\), then \(x\) and \(y\) satisfy the equation

\[
y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).
\]

This equation is called a **two-point** equation of the line.

**Example 5.5.9.** Find an equation of the line through the points \((5, 2)\) and \((-7, 3)\).

**Solution:** We can employ the two-point form here. We get

\[
y - 2 = \frac{3 - 2}{-7 - 5}(x - 5),
\]
which simplifies to
\[ y - 2 = -\frac{1}{12}(x - 5). \]

Most of the forms we have considered so far have one minor flaw: if the line has no slope (i.e., is vertical), we cannot use the form. For example, slope-intercept form \( y = mx + b \) requires that we have \( m \), but if the line is vertical, we don’t. The last form addresses this problem.

**Definition 5.5.2.** The **general form** for an equation of a line is an equation of the form
\[ Ax + By + C = 0. \]

All of the forms we have seen so far can be rewritten in general form. We can rewrite
\[ y = mx + b \]
as \((-m)x + (1)y + (-b) = 0\),
so \( A = -m, B = 1, \) and \( C = -b. \) (Verify this for yourself in the exercises.) We can rewrite
\[ y - y_1 = m(x - x_1) \]
as \((-m)x + (1)y + (mx_1 - y_1) = 0,\)
so \( A = -m, B = 1, \) and \( C = mx_1 - y_1. \) The other forms can be rewritten as well. Notice that for a horizontal line, we have \((0)x + (1)y + (-c) = 0,\) and for a vertical line we have \((1)x + (0)y + (-c) = 0.\)

**Example 5.5.10.** Write an equation of the line through \((4, -2)\) and \((6, 1)\) in general form.

**Solution:** We can use the two-point form to get an equation to start with:
\[ y - (-2) = \frac{1 - (-2)}{6 - 4}(x - 4), \]
or
\[ y + 2 = \frac{3}{2}(x - 4). \]
We need to have the right-hand side be zero in order to have our equation in general form, so we will move everything to the left-hand side.

\[
\begin{align*}
   y + 2 &= \frac{3}{2}(x - 4) \quad \text{Original equation} \\
   -\frac{3}{2}(x - 4) + y + 2 &= -\frac{3}{2}(x - 4) + \frac{3}{2}(x - 4) \quad \text{Adding } -\frac{3}{2}(x - 4) \text{ to both sides} \\
   -\frac{3}{2}x + \frac{3}{2}(4) + y + 2 &= 0 \quad \text{Distributive law and simplification} \\
   -\frac{3}{2}x + 6 + y + 2 &= 0 \quad \text{Simplification} \\
   -\frac{3}{2}x + y + 8 &= 0 \quad \text{Simplification}
\end{align*}
\]

The equation is now in standard form with \( A = -\frac{3}{2}, B = 1, \) and \( C = 8. \)
We will see in the next section that general form is very handy for sketching graphs of lines.

It may seem confusing at first to have so many options for equations of lines. However, there are a couple of points working in your favor. For one thing, the names of the forms say what they are. For example, the point-slope form requires one point and the slope. If you have a point and the slope, this is the form to use.

Secondly, the forms are based on the slope of the line. Did you notice how every time we introduced a new form (except for general form), we went back to the definition of the slope to get started? All of the forms are really just rearrangements of each other!

In summary, we have the following forms of equations of lines. The one you choose will depend on what information you have about the line.

<table>
<thead>
<tr>
<th>Name</th>
<th>Form</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope-intercept</td>
<td>( y = mx + b )</td>
<td>( m ): slope; ( b ): ( y )-intercept</td>
</tr>
<tr>
<td>Point-slope</td>
<td>( y - y_1 = m(x - x_1) )</td>
<td>( m ): slope; ( (x_1, y_1) ): given point</td>
</tr>
<tr>
<td>Two-point</td>
<td>( y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) )</td>
<td>( (x_1, y_1), (x_2, y_2) ): given points</td>
</tr>
<tr>
<td>General</td>
<td>( Ax + By + C = 0 )</td>
<td>( A, B, C ): coefficients</td>
</tr>
<tr>
<td>Horizontal line</td>
<td>( y = c )</td>
<td>( c ): constant</td>
</tr>
<tr>
<td>Vertical line</td>
<td>( x = c )</td>
<td>( c ): constant</td>
</tr>
</tbody>
</table>

*Example 5.5.11.* Find an equation of the line through \((5, -1)\) and \((4, 3)\).

**Solution:** Since we are given two points, we will use the two-point form. Thus

\[
y - (-1) = \frac{3 - (-1)}{4 - 5}(x - 5)
\]

\[
y + 1 = -4(x - 5).
\]

We could rewrite this in any form we choose, but we will stop here.

*Example 5.5.12.* Find an equation of the line through \((4, 2)\) that has slope \(-4\).

**Solution:** This time, we have the slope and one point, so we will use the point-slope form. We have

\[
y - 2 = 4(x - 4).
\]

*Example 5.5.13.* Find an equation of the line through \((-2, 6)\) that is perpendicular to the line \( y = \frac{3}{4}x + 1 \).

**Solution:** We have one point, so we need either another point or the slope. We have no way to find another point, but we do know the slope of a line perpendicular to ours, \(\frac{4}{3}\). [Note:
“Perpendicular to” is symmetric just like “equal to” is symmetric: if line \( a \) is perpendicular to line \( b \), then line \( b \) is perpendicular to line \( a \).] Thus, the slope of our line must be \(-\frac{4}{3}\), so we may now use the point-slope form to finish.

\[
y - 6 = -\frac{4}{3}(x - (-2)) \\
y - 6 = -\frac{4}{3}(x + 2).
\]

\[\square\]

**Example 5.5.14.** Find the slope of the line given by \( 4x + 5y - 2 = 0 \).

**Solution:** If we rewrite this equation in slope-intercept form, we can read off the slope.

\[
4x + 5y - 2 = 0 \\
5y = -4x + 2 \\
y = \frac{1}{5}(-4x + 2) \\
y = -\frac{4}{5}x + \frac{2}{5}.
\]

Thus, the slope of the line is \(-\frac{4}{5}\).

\[\square\]

**Example 5.5.15.** Find the \( y \)-coordinate of the point on the line \( y = -\frac{1}{2}x + 4 \) if its \( x \)-coordinate is 5.

**Solution:** This is another advantage to the slope-intercept form: if we know the \( x \)-coordinate of the point, we can find the \( y \)-coordinate. Whatever \( y \) is, it must satisfy

\[
y = -\frac{1}{2}(5) + 4,
\]

so \( y = -\frac{3}{2} \).

\[\square\]

**Section 5.5 Exercises**

Determine the slope \( m \) and the \( y \)-intercept \( b \) of each line.

1. \( y = 4x + 2 \)
2. \( y = -\frac{3}{4}x + 5 \)
3. \( y = x - 1 \)
4. \( y = \frac{1}{6}x \)
5. \( y = 9 \)
6. \( y = 2x - 4 \)

Find an equation of each line graphed.
Find an equation of the line having the specified properties.


15. Slope $\frac{4}{5}$, through (3, −4).

16. Slope $\frac{1}{8}$, through (−1, −1).

17. Through (2, 5) and (−1, 4).

18. Through (8, −6) and (−2, −3).

19. Slope $\frac{10}{21}$ and y-intercept 0.

20. Through (4, 2) and (8, 2).

21. Through (6, 1) and (6, −9).

22. Parallel to $y = 4x - 3$ and through (2, −5).

23. Parallel to $y = -\frac{5}{7}x + 9$ and through (−9, −2).
24. Perpendicular to \( y = \frac{3}{8} \) and through \((-1, 5)\).

(4, -4).

25. Perpendicular to \(4x + 5y - 3 = 0\) and through \((5, -2)\).

Find the slope \( m \) of each line.

27. \( y + 4 = -\frac{1}{3}(x - 1) \)  
29. \( y = 5x - 1 \)  
31. \( 6x + 5y - 3 = 0 \)

28. \( 3x - 5y + 2 = 0 \)  
30. \( y + 3 = \frac{4}{15}(x + 11) \)  
32. \( 11x - 9y + 14 = 0 \)

Find the \( y \)-coordinate of the point on the given line having the given \( x \)-coordinate.

33. \( y = -7x + 4, x = -2 \)  
36. \( y - 2 = \frac{8}{3}(x + 3), x = 2 \).

34. \( y = -\frac{3}{4}x + \frac{11}{4}, x = 3 \).  
37. \( y = -\frac{15}{16}x + \frac{3}{16}, x = 8 \).

35. \( 4x + 7y - 2 = 0, x = -1 \).  
38. \( 3x + 6y - 12 = 0 \).

5.6 Graphs of lines

In the last section, we took information from graphs of lines to find equations for the lines. In this section, we will use equations of lines to draw their graphs. One key idea to keep in mind is that two points determine a line. That is, if you know two points on the line, you can draw the entire line.

Example 5.6.1. Sketch a graph of the line with equation \( y = 4x + 2 \).

Solution: The equation we are given is in slope-intercept form, so we can see that the \( y \)-intercept is 2 and the slope is 4. We could also find the \( y \)-intercept by letting \( x = 0 \) (since that is the \( x \)-coordinate of any \( y \)-intercept); this gives \( y = 4(0) + 2 = 2 \). Below, we graph the \( y \)-intercept at \((0, 2)\) and then move to the right 1 and up four to get a second point. (Remember, that’s how slope works!) We finish by drawing the line through these two points.
There is an equivalent way to find that second point. If we let $x = 1$, then we see that the corresponding $y$ satisfies $y = 4(1) + 2 = 6$, so the point $(1, 6)$ lies on the graph of the line. Notice that $(1, 6)$ is the other point marked.

□

Example 5.6.2. Sketch the graph of the line with equation $3x + 5y - 30 = 0$.

Solution: Again, if we can find two points that satisfy the given equation, we can plot those and then draw the line joining them. Here is a general principle: for lines, any $x$ we choose will correspond to some $y$ on the line, and any $y$ we choose will correspond to some $x$ on the line. With this in mind, we should choose $x$’s and $y$’s that are convenient for us; that will help us simplify our calculations. If we choose $x = 0$ here, we get

$$3(0) + 5y - 30 = 0,$$

or $5y - 30 = 0$.

To find the corresponding $y$, we must solve this equation.

$$5y - 30 = 0$$
$$5y - 30 + 30 = 0 + 30$$
$$5y = 30$$
$$\frac{1}{5}(5y) = \frac{1}{5}(30)$$
$$y = 6.$$

Thus, the point $(0, 6)$ is on the graph of the given line. Notice that setting $x = 0$ gave us the $y$-intercept, as it should.

Now we need a second point, so we will set $y$ equal to 0; we have

$$3x + 5(0) - 30 = 0,$$

or $3x - 30 = 0$. 
We need to solve this to find the corresponding value of $x$.

\[
3x - 30 = 0 \\
3x - 30 + 30 = 0 + 30 \\
x = 30
\]

\[
\frac{1}{3}(3x) = \frac{1}{3}(30) \\
x = 10.
\]

Thus, the point $(10, 0)$ also lies on the line. We plot these points below and join them with a line.

Just as the point at which the line crosses the $y$-axis is called the $y$-intercept, the point at which the line crosses the $x$-axis is called the $x$-intercept. Its $y$-coordinate is 0 at that point. In the example above, the $x$-intercept of the line was $(10, 0)$ or just 10. Often, the $x$- and $y$-intercepts of a line are very convenient points to use in graphing the line.

**Example 5.6.3.** Sketch a graph of the line with equation $4x + 6y + 12 = 0$.

**Solution:** To find the $y$-intercept, we set $x = 0$ and solve $6y + 12 = 0$ for $y$, giving $y = -2$. To find the $x$-intercept, we set $y = 0$ and solve $4x + 12 = 0$ for $x$, giving $x = -3$. These two points suffice to graph the line.

![Figure 5.38: The line $4x + 6y + 12 = 0$](image)

**Example 5.6.4.** Sketch the graph of the line with equation $y = -\frac{2}{5}x + 3$.

**Solution:** We are given that the $y$-intercept is 3 and that the slope is $-\frac{2}{5}$. Thus, we can plot the point $(0, 3)$, and then move to the right 5 units and down 2 units to find the second point. After that, we can just draw the line through those two points.
Example 5.6.5. Sketch the graph of the line with equation \( x = 4 \).

**Solution:** This is a vertical line, and its \( x \)-intercept is 4.

\[
\begin{align*}
\text{Figure 5.40: The line } x &= 4 \\
\end{align*}
\]

Example 5.6.6. Sketch the graph of the line with equation \( y + 2 = \frac{1}{2}(x - 3) \).

**Solution:** If we set \( x = 0 \), we have \( y + 2 = \frac{1}{2}(-3) \), so \( y = -\frac{3}{2} - 2 = -\frac{7}{2} \). If we set \( x = 3 \), we have \( y + 2 = 0 \), so \( y = -2 \). (We chose \( x = 3 \) because that made the right-hand side 0, which is easy to work with.) Thus, the two points we have are \((0, -7/2)\) and \((3, -2)\).
Figure 5.41: The line \( y + 2 = \frac{1}{2}(x - 3) \)

Section 5.6 Exercises

Find the \( x \)- and \( y \)-intercepts of each line.

1. \( y = \frac{2}{3}x + 5 \)  
2. \( y - 3 = 5(x + 2) \)  
3. \( 4x + 5y - 10 = 0 \)  
4. \( y - 3 = \frac{3}{4}(x - 8) \)  
5. \( 2x + 12y - 6 = 0 \)

Sketch the graph of each line.

7. \( y = \frac{2}{3}x + 5 \)  
8. \( y - 3 = 5(x + 2) \)  
9. \( 4x + 5y - 10 = 0 \)  
10. \( y - 3 = \frac{3}{4}(x - 8) \)  
11. \( 2x + 12y - 6 = 0 \)  
12. \( 3x + 5y + 60 = 0 \)  
13. \( y = 2x + 8 \)  
14. \( 4x - 6y + 24 = 0 \)  
15. \( y + 1 = \frac{5}{2}(x - 4) \)  
16. \( -3x - 4y + 9 = 0 \)  
17. \( y + 6 = -\frac{4}{5}x \)  
18. \( y = \frac{8}{3}x - 2 \)  
19. \( x - 3 = 0 \)  
20. \( y - 3 = 2x \)  
21. \( y = -\frac{5}{2} \)  
22. \( y = 6x \)  
23. \( -2x + 4y - 6 = 0 \)  
24. \( x + 4y - 1 = 0 \)
Chapter 6

Functions

Functions are very common and very useful mathematical objects. Scientists in all fields use them to describe the behavior of things they study, as do professionals in many other fields. Functions are used to describe the motion of a particle, the size of a population as it changes over time, the amount of a substance created in a chemical reaction, the amount of seed it takes to plant fields of different sizes, and so on.

In this Unit, we introduce the idea of a function and explore some of the more common functions.

6.1 Definitions

Example 6.1.1. Consider the table below. It shows how the area of a rectangle of height 4 changes as its length changes; that is, how the area depends on the length.

<table>
<thead>
<tr>
<th>Length</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
</tr>
</tbody>
</table>

Notice how each length corresponds to just one area. This is the hallmark of a function.

Definition 6.1.1. Suppose that we have two sets (collections) of objects. A function is a rule that assigns exactly one object from the second set to each object from the first set. (It is possible that an object from the second set is used more than once or not at all.) We say that the objects in the second set are a function of the objects in the first set.

In Example 6.1.1, we have the area of the rectangle as a function of its length. The “first set” was the set of possible lengths of the rectangle, and the “second set” was the set of possible areas.

In this course, we primarily will be interested in collections of numbers rather than just “objects”, but there are times when it makes sense to consider more generic things.
**Example 6.1.2.** Every person in your class has a name and a height. Here is a table that lists names and heights for a fictional class.

<table>
<thead>
<tr>
<th>Name</th>
<th>Sarah</th>
<th>Kim</th>
<th>Ken</th>
<th>Jill</th>
<th>Roy</th>
<th>Joe</th>
<th>Robert</th>
<th>Greg</th>
<th>Ellen</th>
<th>Wayne</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (inches)</td>
<td>59</td>
<td>64</td>
<td>73</td>
<td>61</td>
<td>70</td>
<td>66</td>
<td>75</td>
<td>75</td>
<td>65</td>
<td>73</td>
</tr>
</tbody>
</table>

Notice that each person has exactly one height, so height is a function of the person. That is, our “first set” is made up of students, our “second set” is made up of heights, and every person is assigned exactly one height. Now let’s present the table in a slightly different way.

<table>
<thead>
<tr>
<th>Height (inches)</th>
<th>59</th>
<th>61</th>
<th>64</th>
<th>65</th>
<th>66</th>
<th>68</th>
<th>70</th>
<th>73</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name</td>
<td>Sarah</td>
<td>Jill</td>
<td>Kim</td>
<td>Ellen</td>
<td>Joe</td>
<td>Debbie</td>
<td>Roy</td>
<td>Wayne</td>
<td>Greg</td>
</tr>
</tbody>
</table>

Since some heights are paired with more than one person, the person is not a function of the height.

It is becoming somewhat unwieldy to keep talking about the “first set” and the “second set,” so let’s introduce some terminology to make our lives easier.

**Definition 6.1.2.** In the definition of a function, the “first set” is called the **domain** of the function, and the “second set” is called the **range** of the function. Members of the domain are also sometimes referred to as **input**, and members of the range are sometimes referred to as **output**.

**Example 6.1.3.** In Example 6.1.1 of this section, the domain was the set of possible lengths of the rectangle – which means any positive numbers – and the range was the set of possible areas – also any positive numbers. Thus, both the domain and range are equal to \( \{ x \mid x > 0 \} \).

**Example 6.1.4.** In the second example, the domain is made up of the people and the range is made up of the heights. That is, the domain is \{Sarah, Jill, Kim, Ellen, Joe, Debbie, Roy, Mark, Ken, Robert\} and the range is \{59, 61, 64, 65, 66, 68, 70, 73, 75\}.

One way to look at a function is as a reliable machine. If you put a 5 into the machine and it gives you back a 12, then when you come back two days later and give it a 5 again, it will give you a 12 again. In Example 6.1.2 above, the second table does not give a reliable relationship. Sometimes it will give you Ken for a height of 73 inches, and sometimes it will give you Wayne!
Example 6.1.5. The area of a circle of radius \( r \) is \( \pi r^2 \), where \( \pi \) is the famous number approximately equal to 3.14159. Given a particular radius, like 4 cm, the value of

\[
\pi r^2 = \pi (4)^2 = 16\pi
\]

will always be the same, so this is reliable. That is, the area of a circle is a function of its radius.

Example 6.1.6. Let’s define the function with input \( x \) and output \( \pm x \). The symbol \( \pm \) is pronounced “plus or minus” and means that we have two values. Thus, \( f(2) = 2 \) or \( f(2) = -2 \). This can’t be a function because it doesn’t always give us the same result when we give it 2!

Example 6.1.7. Let \( D = \{1, 2, 3, 4, 5, 6\} \). The function described by the table below does not have domain \( D \) since it is not defined for \( x = 6 \). That is, even though it is a function, it is not a function with domain \( D \).

<table>
<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>2</td>
<td>3</td>
<td>-2</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Instead, its domain is \( \{1, 2, 3, 4, 5\} \).

Example 6.1.8. Let \( D = \{1, 2, 3, 4, 5, 6\} \) and \( R = \{1, 2, 3, 4, 5, 6, \ldots\} \). (That is, \( R \) is the set of natural numbers.) Define a function with domain \( D \) so that for an input of \( x \), then output is \( x + 3 \). There is nothing inherent in the formula for the function that prevents us from substituting 7 for \( x \), but we have specified that 7 is not in the domain of the function.

Notice that the range of this function includes a lot of values that the function doesn’t really need. For example, there is no input in the domain of this function that corresponds to an output of 1. (It’s true that \(-2 + 3 = 1\), but \(-2\) is not in the domain!)

Example 6.1.9. The function in this example takes a real number \( x \) as its input and returns the real number \( 7x - 1 \) as its output. Since any real number \( x \) can be multiplied by 7, and 1 can be subtracted from that result (no matter what it is!), the domain is the set of all real numbers.
Example 6.1.10. The function in this example will take a real number \( x \) as its input, and give back the real number \( \frac{x+1}{x} \) as its output. However, there is a potential problem: if \( x = 0 \), then \( \frac{x+1}{x} \) is not defined! This means that \( x = 0 \) cannot be part of the domain of this function. On the other hand, if \( x \neq 0 \), then \( \frac{x+1}{x} \) is a real number, so \( x = 0 \) is the only number excluded from the domain. Thus, the domain is the set of all real numbers except 0.

Example 6.1.11. What is the domain of the function that gives an output of \( \sqrt{x} \) for an input of \( x \)?

Solution: Remember that \( \sqrt{x} \) means the positive number whose square is \( x \). If \( x \) is negative, there are no numbers whose square is \( x \), so \( x \) is not allowed to be negative. Therefore, the domain of this function is \( \{x | x \geq 0\} \).

Example 6.1.12. What is the domain of the function that gives an output of \( \frac{x+1}{x^2 + x - 6} \) for an input of \( x \)?

Solution: Since we have a denominator, we have the possibility that some value of \( x \) will make the denominator 0; such values of \( x \) must be excluded. In order to see what those values are, we first need to factor the denominator (which, luckily, we learned how to do in Chapter 4!).

\[
x^2 + x - 6 = (x + 3)(x - 2),
\]

so the values of \( x \) we cannot allow are \( x = -3 \) and \( x = 2 \). Therefore, the domain is \( \{x | x \neq 2, -3\} \).

In a later course, you may see a more formal definition of function. If you do, compare it to this one until you can convince yourself that they mean the same thing.

Section 6.1 Exercises

Determine which of the following represent functions with the given domain and range. If one is not a function, state what part(s) of the definition it fails.

1. Domain: \( \{1, 2, 3, 4, 5\} \). Range: \( \{1, 2, 3, 4, 5\} \). The output corresponding to an input of \( x \) is \( x \).

2. Domain: \( \{1, 2, 3, 4, 5\} \). Range: \( \{1, 2, 3, 4, 5\} \). The output corresponding to an input of \( x \) is \( x + 1 \).

3. Domain: the set of all integers. Range: the set of all integers. The output corresponding to an input of \( x \) is \( \frac{x}{x^2 + 1} \).
4. Domain: the set of all natural numbers. Range: the set of all integers. The output corresponding to an input of $x$ is $-x$.

5. Domain: the set of all natural numbers. Range: the set of all natural numbers. The output corresponding to an input of $x$ is $-x$.

6. Domain: the set of all rational numbers. Range: the set of all rational numbers. The output corresponding to an input of $x$ is $\sqrt{x}$.

7. Domain: the set of all real numbers. Range: the set of all real numbers. The output corresponding to an input of $x$ is $\sqrt{x}$.

8. Domain: the set of all natural numbers. Range: the set of all rational numbers. The output corresponding to an input of $x$ is $\frac{x+1}{x}$.

9. Domain: the set of all real numbers. Range: the set of all real numbers. The output corresponding to an input of $x$ is $\frac{1}{x}$.

10. Domain: the set of all real numbers. Range: the set of all real numbers. The output corresponding to an input of $x$ is $\frac{x}{x^2+1}$.

11. Domain: the set of all real numbers. Range: the set of all rational numbers. The output corresponding to an input of $x$ is $x - 2$.

12. Domain: the set of all real numbers. Range: the set of all real numbers. The output corresponding to an input of $x$ is $x$.

Each function below is described in terms of input and output. For the inputs specified, give the corresponding output.


15. Input: $x$. Output: $\frac{x}{2}$. Inputs: $-2, 5, 12, 99$.

16. Input: $x$. Output: $8x$. Inputs: $-\frac{1}{4}, \frac{1}{2}, 4, 6$.

17. Input: $s$. Output: $\frac{s + 1}{s^2 + 1}$. Inputs: $-2, -1, 5, 6$.

18. Input: $x$. Output: $x^2 + 3x - 1$. Inputs: $-2, 0, 4, 5$.

19. Input: $x$. Output: $\frac{1}{x}$. Inputs: $-4, 1, \frac{1}{3}, \frac{1}{2}, 12$.

20. Input: $x$. Output: $3x - 5$. Inputs: $-4, 5, \frac{5}{3}, x + 1, \frac{1}{x}$.

21. Input: $t$. Output: $t^4 + t^2 + 1$. Inputs: $-1, 1, 0, (t - 3)$.
22. Input: \( t \). Output: \( \sqrt{t + 1} - t \). Inputs: \( 2, 5, t^2 + 1, 15 \).

23. Input: \( x \). Output: \( \sqrt{x^2} \). Inputs: \(-2, 2, -5, 5, -10, 10 \).

<table>
<thead>
<tr>
<th>input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>output</td>
<td>2</td>
<td>3</td>
<td>-2</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>16</td>
<td>-3</td>
<td>11</td>
<td>2</td>
</tr>
</tbody>
</table>

Inputs: 2, 5, 7, 8.

Find the domain of each function described by the given rule. In all cases, the input is \( x \) and the output is all that is given.

25. \( 7x + 12 \)  
26. \( 3x + 2 \)  
27. \( 4x^2 - 3x - 11 \)  
28. \( \frac{3x + 1}{4} \)  
29. \( \frac{4}{3x + 1} \)

30. \( \sqrt{x + 1} \)
31. \( \sqrt{3x + 5} \)
32. \( \frac{4x}{5x + 10} \)
33. \( \frac{3x + 2}{3x - 2} \)
34. \( \frac{-x}{2 - x} \)
35. \( \frac{10x}{x^2 - 6x + 8} \)
36. \( \frac{2x - 4}{x^2 - 9} \)

### 6.2 Notation

It is usually pretty inconvenient to make a table to represent a function. For example, we can’t make a complete table for the function in Example 6.1.1 because there are infinitely many values! Most of the time we have a formula to describe the function, and we also give the function a name so we can refer to it easily. “The function that gives the area of a rectangle of height 4 in terms of its length” is rather a lot to say, and so is “the function that gives an output of \( 4x \) for an input of \( x \).”

When we can, we give functions names that will remind us what the functions are (or do); for example, we might call “the function that gives the area of a rectangle of height 4 in terms of its length” just \( A \), for area. As long as we stay in the context of that problem, any time we use the letter \( A \), we mean that function. Once we finish with that problem, we free up the letter \( A \) for a different function.

If we have a generic function or one that doesn’t have a special meaning to us (like “area”), we will typically use the letter \( f \), for “function.” Notice also that function names are italicized. If we need more than one such function, we usually add letters alphabetically from there, so we might call our second function \( g \) and our third function \( h \).

**Example 6.2.1.** The table below represents the function \( f \) with domain \{1, 2, 3, 4, 5\} and range \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

<table>
<thead>
<tr>
<th>( f )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>
Now we can ask, what is the output from \( f \) when the input is 2? (It is 3.) This is still a little unwieldy, though, so we are going to introduce a further simplification.

**Definition 6.2.1.** If \( f \) is the name of a function and \( x \) is a member of the domain of \( f \), then \( f(x) \) represents the output from \( f \) when the input is \( x \). The symbol \( f(x) \) is pronounced, “\( f \) of \( x \)”. We also say that \( f(x) \) is “\( f \) evaluated at \( x \)” or “the value of \( f \) at \( x \).”

**CAUTION:** Even though \( f(x) \) looks like multiplication, it is not. You will have to be alert for context to decide whether multiplication or function evaluation is intended.

This allows us easily to describe \( f \) in terms of a formula. For example, if we want \( f \) to assign the output \( 3x - 1 \) to the input \( x \), we just have to say, “Define \( f \) by \( f(x) = 3x - 1 \).” Then if we want to know \( f(2) \), we just substitute 2 for \( x \) in the formula. Thus,

\[
f(2) = 3(2) - 1 \\
= 6 - 1 \\
= 5.
\]

Also, we write \( f(x) \) to indicate the value of \( f \) at \( x \), no matter what \( x \) is. This is the nature of the variable \( x \) (or any other variable) – it is just a placeholder until you know what value you want it to take.

This also means that the name of the variable is irrelevant; I can call it anything I want. The letter \( x \) is customary, but \( t \) is also used when the variable represents time, \( i \) is often used if the variable has to be an integer, and so on. The point is, don’t get hung up on whether we write \( f(x) \) or \( f(t) \); we are really talking about the same thing.

**Example 6.2.2.** Define the function \( f \) by \( f(x) = x^2 - 3x + 1 \). Evaluate \( f(1) \), \( f(-1) \), \( f(12) \), \( f(0) \), \( f \left( \frac{2}{5} \right) \), and \( f(t) \).

**Solution:** The notation \( f(x) = x^2 - 3x + 1 \) tells us that the input \( x \) is assigned to the output \( x^2 - 3x + 1 \). That is, whatever \( x \) we are given as input, we first square it, then subtract 3 times the input from that, and then add 1.

1.

\[
f(1) = 1^2 - 3(1) + 1 \\
= 1 - 3 + 1 \\
= -1.
\]

2.

\[
f(-1) = (-1)^2 - 3(-1) + 1 \\
= 1 + 3 + 1 \\
= 5.
\]
3. 

\[ f(12) = 12^2 - 3(12) + 1 \]
\[ = 144 - 36 + 1 \]
\[ = 109. \]

4. 

\[ f(0) = 0^2 - 3(0) + 1 \]
\[ = 0 - 0 + 1 \]
\[ = 1. \]

5. 

\[ f \left( \frac{2}{5} \right) = \left( \frac{2}{5} \right)^2 - 3 \left( \frac{2}{5} \right) + 1 \]
\[ = \frac{2^2}{5^2} - \frac{3 \cdot 2}{5} + 1 \]
\[ = \frac{4}{25} - \frac{6}{5} + 1 \]
\[ = \frac{4}{25} - \frac{30}{25} + \frac{25}{25} \]
\[ = \frac{4 - 30 + 25}{25} \]
\[ = \frac{-1}{25}. \]

6. \( f(t) = t^2 - 3t + 1. \) Notice that this looks just like \( f(x), \) except that we have a \( t \) instead of an \( x. \)

\[ \square \]

We must take the instructions literally. \textit{No matter what} value is assigned to \( x, \) we substitute \textit{exactly} that value into the formula in place of every \( x \) that appears.

\textbf{Example 6.2.3.} Define \( f \) by \( f(x) = 4x + 1. \) Evaluate \( f(t), f(t+1), f(♣), \) and \( f(\text{Encyclopedia Bizzarica}). \)

\textbf{Solution:}

1. \( f(t) = 4(t) + 1 = 4t + 1. \) All we did was change \( x \) to \( t. \)

2. 

\[ f(t+1) = 4(t+1) + 1 \]
\[ = 4t + 4 + 1 \]
\[ = 4t + 5. \]

The rule says that \textit{no matter what value} we have for the input, we first multiply it by 4 and then add 1 to the result.
3. \( f(♣) = 4♣ + 1. \)

4. \( f(\text{Encyclopedia Bizzarica}) = 4(\text{Encyclopedia Bizzarica}) + 1. \)

\[ \square \]

**Example 6.2.4.** Define \( f \) by \( f(x) = \sqrt{x + 2} \). Find the domain of \( f \) and evaluate \( f(2), f(-1), f(7), \) and \( f(3) \).

**Solution:** Since we are not allowed to take the square root of a negative number, we must have \( x + 2 \geq 0 \), or \( x \geq -2 \). Thus, the domain of \( f \) is \( \{x | x \geq -2\} \).

1. \( f(2) = \sqrt{2 + 2} = \sqrt{4} = 2. \)
2. \( f(-1) = \sqrt{-1 + 2} = \sqrt{1} = 1. \)
3. \( f(7) = \sqrt{7 + 2} = \sqrt{9} = 3. \)
4. \( f(3) = \sqrt{3 + 2} = \sqrt{5}. \)

This is an example of a **radical function** since the output of the function is a radical expression.

\[ \square \]

**Example 6.2.5.** Define \( f \) by \( f(x) = 4x + 3 \). Find the domain of \( f \) and evaluate \( f(4), f(-7), f(2), \) and \( f(x + 1) \).

**Solution:** The domain of \( f \) is the set of all real numbers.

1. \( f(4) = 4(4) + 3 = 19. \)
2. \( f(-7) = 4(-7) + 3 = -25. \)
3. \( f(2) = 4(2) + 3 = 11. \)
4. \( f(x + 1) = 4(x + 1) + 3 = 4x + 4 + 3 = 4x + 7. \)

This function is an example of a **linear function**. Notice that if we replace \( f(x) \) with \( y \), we have \( y = 4x + 3 \), which is the slope-intercept form of an equation of a line.

\[ \square \]

**Example 6.2.6.** Define \( f \) by \( f(x) = x^2 - 3x + 1 \). Find the domain of \( f \) and evaluate \( f(-3), f(t), f(4), \) and \( f(x - 2) \).

**Solution:** The domain of \( f \) is the set of all real numbers.

1. \( f(-3) = (-3)^2 - 3(-3) + 1 = 19. \)
2. \( f(t) = t^2 - 3t + 1. \)
3. \( f(4) = 4^2 - 3(4) + 1 = 5. \)
4. 

\[ f(x - 2) = (x - 2)^2 - 3(x - 2) + 1 = x^2 - 4x + 4 - 3x + 6 + 1 = x^2 - 7x + 11. \]

This is an example of a polynomial function since the output of the function is a polynomial expression. Notice that linear functions are also polynomial functions.

\[ \square \]

Example 6.2.7. Define \( f \) by \( f(x) = \frac{x + 2}{3x - 4} \). Find the domain of \( f \) and evaluate \( f(-1), f(2), f(5), \) and \( f(x - 3) \).

**Solution:** We are not allowed to have the denominator equal zero, so we need \( 3x - 4 \neq 0 \). Solving \( 3x - 4 = 0 \) gives \( x = \frac{4}{3} \), so the domain is \( \{x | x \neq \frac{4}{3}\} \).

1. \( f(-1) = \frac{-1 + 2}{3(-1) - 4} = \frac{-1}{7} \).
2. \( f(2) = \frac{2 + 2}{3(2) - 4} = 2 \).
3. \( f(5) = \frac{5 + 2}{3(5) - 4} = \frac{7}{11} \).
4. \( f(x - 3) = \frac{(x - 3) + 2}{3(x - 3) - 4} = \frac{x - 1}{3x - 9 - 4} = \frac{x - 1}{3x - 13} \).

This is an example of a rational function since the output of the function is a rational expression.

\[ \square \]

We can use the idea of function evaluation to draw a graph of the function; often, it is easier to interpret the behavior of a function from its graph than from its formula. The idea is that the input is the \( x \)-coordinate and the output is the \( y \)-coordinate. Thus, a point on the graph of \( f \) looks like \((x, f(x))\).

We have already seen a number of graphs of linear functions (their graphs are just lines!), although we did not refer to them that way at the time.

Our approach to graphing functions is to compute a few points on the graph and then join them with a smooth curve. Of course, since we are only choosing a few points, it is possible that our smooth curve misses some important features of the graph, but we should be able to obtain fairly accurate representations.

**Example 6.2.8.** Sketch the graph of \( f(x) = x^2 \).

**Solution:** We first make a table of representative values of the function. When we plot these points, we will see an outline of the graph of the function, which we can then fill in with a smooth curve.
These points are plotted below.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
y & 9 & 4 & 1 & 0 & 1 & 4 & 9 \\
\hline
\end{array}
\]

Figure 6.1: Some points on the graph of \( f(x) = x^2 \)

Now we join these points with a smooth curve.

Figure 6.2: \( f(x) = x^2 \)

**Example 6.2.9.** Sketch the graph of \( f(x) = \sqrt{x + 1} \).

**Solution:** We again make a table of representative values. Since we are choosing the values of \( x \), we may as well choose some convenient ones. Also, the domain of \( f \) is \( \{x | x \geq -1\} \).

\[
\begin{array}{|c|c|c|c|}
\hline
x & -1 & 0 & 3 \\
y & 0 & 1 & 8 \\
\hline
\end{array}
\]

These points are plotted below.
Figure 6.3: Some points on the graph of $f(x) = \sqrt{x + 1}$

Now we join these points with a smooth curve.

Figure 6.4: $f(x) = \sqrt{x + 1}$

Example 6.2.10. Sketch the graph of $f(x) = \frac{1}{x}$.

Solution: We again make a table of representative values. The domain of $f$ is $\{x | x \neq 0\}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$-\frac{1}{2}$</th>
<th>$-\frac{1}{3}$</th>
<th>$\frac{1}{3}$</th>
<th>$\frac{1}{2}$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$-3$</td>
<td>$3$</td>
<td>$2$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

These points are plotted below.
Figure 6.5: Some points on the graph of $f(x) = \frac{1}{x}$

Now we join these points with a smooth curve.

Figure 6.6: $f(x) = \frac{1}{x}$

Since $x = 0$ is not in the domain of $f$, the graph can't actually touch the $y$-axis. Thus, it is in two pieces – one to the left of the $y$-axis, and one to the right of the $y$-axis.

You can also use your graphing calculator to graph functions.

Section 6.2 Exercises

Find the domain of each function.

1. $f(x) = 7x + 2$
2. $f(x) = \frac{3}{4x + 5}$
3. $f(x) = \sqrt{4 - 2x}$
4. $f(x) = \frac{x + 2}{x^2 - 1}$
5. $f(x) = \frac{1}{x^2 - 3x - 10}$
6. $f(x) = 5x^3 + 7x - 11$. 
Identify each function as linear, polynomial, rational, radical, or none of these.

1. \( f(x) = x^6 + 5x - 3 \)
2. \( f(x) = \frac{3x^2 + 1}{5x - 2} \)
3. \( f(x) = \sqrt{x^3 - 3x + 1} \)
4. \( f(x) = 7x - 1 \)
5. \( f(x) = x^\frac{5}{3} \)
6. \( f(x) = \sqrt{\frac{x + 2}{x - 1}} \)

Evaluate each function as indicated.

7. \( f(x) = 7x + 2 \). Evaluate \( f(-3) \).
8. \( f(x) = \frac{3}{4x+5} \). Evaluate \( f(2) \).
9. \( f(x) = \sqrt{4-2x} \). Evaluate \( f(-8) \).
10. \( f(x) = \frac{x+2}{x^2-1} \). Evaluate \( f(2) \).
11. \( f(x) = \frac{1}{x^2-3x-10} \). Evaluate \( f(1) \).
12. \( f(x) = 5x^3 + 7x - 11 \). Evaluate \( f\left(\frac{1}{3}\right) \).
13. \( f(x) = x^6 + 5x - 3 \). Evaluate \( f(-1) \).
14. \( f(x) = \frac{3x^2 + 1}{5x - 2} \). Evaluate \( f\left(\frac{1}{7}\right) \).
15. \( f(x) = \sqrt{x^3 - 3x + 1} \). Evaluate \( f(2) \).
16. \( f(x) = 7x - 1 \). Evaluate \( f(-5) \) and \( f(2x - 3) \).
17. \( f(x) = x^\frac{5}{3} \). Evaluate \( f(64) \).
18. \( f(x) = \frac{x+2}{x^2-1} \). Evaluate \( f(2) \) and \( f(-x) \).
19. \( f(x) = x^2 \). Evaluate \( f(-5) \), \( f(5) \), \( f(3x) \), and \( f(x-1) \).
20. \( f(x) = \frac{3}{2x} \). Evaluate \( f\left(\frac{2}{3}\right) \) and \( f(x-1) \).
21. \( f(x) = \frac{\sqrt{x}}{\sqrt{x+1}} \). Evaluate \( f(0) \) and \( f(3) \).
22. \( f(x) = \sqrt{x^2} \). Evaluate \( f(8), f(4) \), and \( f(9) \).

Sketch the graph of each function.
6.3 Piecewise-Defined Functions

In this section we consider another kind of function called a **piecewise-defined function**. These functions occur frequently; the idea is that the rule for the function changes depending on the values. Consider the following examples.

**Example 6.3.1.** Some long distance telephone plans charge a monthly fee that covers all long-distance calls up to a certain number of minutes, and then charge by the minute thereafter. Let’s let \( L \) denote the long distance cost function, and we will describe \( L \) in terms of the variable \( t \) (for time).

Suppose that the plan is $19.95 per month for up to 300 minutes of long-distance calls, and additional minutes are 5 cents each. Then we can describe \( L \) in two pieces as follows:

\[
L(t) = \begin{cases} 
$19.95 & \text{if } t \leq 300 \\
$19.95 + 0.05(t - 300) & \text{if } t > 300 
\end{cases}
\]

where \( t \) is in minutes. This is interpreted just as its written: if \( t \) (the number of minutes of long distance used) is less than or equal to 300, then the charge is just $19.95. If \( t \) is more than 300 minutes, then we have to pay the $19.95 plus an additional 5 cents (or $0.05) per minute over the 300 minutes. If \( t \) is the total number of minutes, then the number of minutes exceeding 300 is \( t - 300 \).

Thus,

\[
L(50) = $19.95 \\
L(100) = $19.95 \\
L(299) = $19.95 \\
L(300) = $19.95 + 0.05(300 - 300) = $19.95 \\
L(375) = $19.95 + 0.05(375 - 300) = $19.95 + 18.75 = $38.70,
\]

and so on.

The previous example illustrates the way piecewise functions are written. The curly brace indicates which lines are part of the function. With each line, there is a description of how the function behaves for certain values of the variable; above, there were two sets of values to work with: those less than 300 minutes, and those greater than or equal to 300 minutes.

Here is another famous (and important!) example of a piecewise-defined function.

**Example 6.3.2.** The **absolute value** function is defined by

\[
|x| = \begin{cases} 
x & \text{if } x \geq 0 \\
-x & \text{if } x < 0
\end{cases}
\]

Let’s calculate some values of this function.
There are several things to notice about this function. First, you can see from the examples (as well as the definition) that if a number is positive (or zero), the absolute value function does nothing to it. If a number is negative, the absolute value function changes the sign of the number so that it is positive. In a sense, the absolute value function tells you the "size" of a number, without regard to sign.

Also, the vertical bars (|) are grouping symbols; if something is between them (like $5 - 8$ was in part (k) of the last example), the operations inside must be carried out before other operations.

The function $\sqrt{x^2}$ does exactly the same thing as the absolute value function. For example, $\sqrt{(-3)^2} = \sqrt{9} = 3$, and $|-3| = 3$, too. Thus,

$$\sqrt{x^2} = |x|.$$
Figure 6.7: The absolute value function

Notice that to the right of the $y$-axis, the graph looks just like the graph of $y = x$, and to the left of the $y$-axis, it looks like the graph of $y = -x$. (Why is that?)

□

Example 6.3.3. Define the function $f$ by

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0. \end{cases}$$

Then

$$f(-4) = -2$$
$$f(-3) = -2$$
$$f(-0.003) = -2$$
$$f(0) = 1$$
$$f(1) = 1$$
$$f(40) = 40$$
$$f(165234) = 165234.$$  

In each case, we examine the value of $x$ to see which of the three categories it falls into, and evaluate $f$ based on that.

□
Example 6.3.4. The cost of mailing a package depends on how heavy the package is. The function $f$ describes this cost.

\[
f(x) = \begin{cases} 
  \$3.95 & \text{if } x \text{ is under 2 pounds} \\
  \$5.25 & \text{if } x \text{ is between 2 and 4 pounds} \\
  \$6.50 & \text{if } x \text{ is between 4 and 6 pounds} \\
  $1.05x & \text{if } x \text{ is more than 6 pounds.}
\end{cases}
\]

A package weighing 3.5 pounds would cost $f(3.5) = \$5.25$. A package weighing 4.8 pounds would cost $f(4.8) = \$6.50$. A package weighing 12 pounds would cost $\$1.05(12) = \$12.60$.

\[\square\]

Section 6.3 Exercises

Evaluate each function as indicated.

1. $f(x) = \begin{cases} 
  -x & \text{if } x < 2 \\
  x + 1 & \text{if } x \geq 2
\end{cases}$ at $x = -5, -2, 1, 2, 4$.

2. $f(x) = \begin{cases} 
  x^2 & \text{if } x \leq 4 \\
  x^2 + 1 & \text{if } x > 4
\end{cases}$ at $x = 0, 3, 4, 6$.

3. $f(x) = \begin{cases} 
  2x + 1 & \text{if } x < -1 \\
  -x - 2 & \text{if } x \geq -1
\end{cases}$ at $x = -3, -2, -1, 2$.

4. $f(x) = \begin{cases} 
  \sqrt{-x} & \text{if } x < 0 \\
  \sqrt{x} & \text{if } x \geq 0
\end{cases}$ at $x = -4, -1, 0, 9$.

5. $f(x) = |2x - 3|$ at $x = -2, 1, 4, 7$.

6. $f(x) = |-x|$ at $x = -5, -7, 2, 4$.

7. $f(x) = |x| - 2$ at $x = -2, 0, 2, 5$.

8. $f(x) = |5 - x|$ at $x = -5, -2, 5, 11$.

Sketch the graph of each function.

1. $f(x) = |x - 1|$  \hspace{1cm} 3. $f(x) = |2x|$  \hspace{1cm} 5. $f(x) = |x| + 1$

2. $f(x) = |x + 2|$  \hspace{1cm} 4. $f(x) = |x - 3|$  \hspace{1cm} 6. $f(x) = |x| - 3$

7. A rental car company charges a daily rate of $24.95 plus 35 cents per mile beyond 200 miles. (The first 200 miles are “free”.) Express the cost of renting a car for one day in terms of the number of miles driven. Then determine the costs for driving 50, 150, 250, and 350 miles.
8. Find a tax table (available at http://www.irs.ustreas.gov) and find the tax corresponding to your income. (The tax tables are a gigantic piecewise-defined function.)

9. A mail-order book company advertises shipping rates of $5 for up to 6 books, and then $0.75 for each additional book.
   
   (a) Determine a reasonable domain for this function.
   (b) Write a description of this function. (Note that it is piece-wise defined.)
   (c) Find the shipping cost for (i) 3 books and (ii) 12 books.

6.4 Operations on Functions

Just as we can add, subtract, multiply, and divide numbers, we can add, subtract, multiply, and divide functions.

**Definition 6.4.1.** If $f$ and $g$ are functions, we define the **sum** of $f$ and $g$ to be the function $f + g$, where

$$(f + g)(x) = f(x) + g(x)$$

for any real number $x$ that is in the domain of both $f$ and $g$. Similarly, we define the **difference** $f - g$ by

$$(f - g)(x) = f(x) - g(x)$$

and the **product** $f \cdot g$ by

$$(f \cdot g)(x) = f(x)g(x)$$

for any real number $x$ that is in the domain of both $f$ and $g$.

**Remember:** the notation $f(x)$ looks like multiplication, but is really function evaluation when $f$ is a function and $x$ is a real number. This is even more deceptive when you have $(f + g)(x)$, but this is still function evaluation: we are evaluating the function called $f + g$ at the real number $x$.

**Example 6.4.1.** If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, then

$$(f + g)(x) = f(x) + g(x) = \sqrt{x} + x + 1.$$  

Since we must compute both $f(x)$ and $g(x)$, $x$ must be in the domains of both $f$ and $g$. Thus,

$$(f + g)(4) = \sqrt{4} + 4 + 1 = 2 + 5 = 7,$$

while $(f + g)(-3)$ is undefined since $f(-3)$ is undefined.

Also,

$$(f - g)(x) = f(x) - g(x) = \sqrt{x} - (x + 1) = \sqrt{x} - x - 1$$

and

$$(f \cdot g)(x) = f(x)g(x) = \sqrt{x}(x + 1).$$
Example 6.4.2. Let \( f(x) = 2x + 1 \) and \( g(x) = x - 2 \). Then \((f \cdot g)(x) = (2x + 1)(x - 2)\). Also, 
\[
(f \cdot g)(-1) = (2(-1) + 1)(-1 - 2) = 3.
\]

\[
(f - g)(2) = f(2) - g(2) = (2(2) + 1) - (2 - 2) = 5.
\]

\[
(f + g)(4) = f(4) + g(4) = (2(4) + 1) + (4 - 2) = 11.
\]

The domains of all of these functions are the same; they are all the set of real numbers.

\(\square\)

You may have noticed that we omitted division from our definition above. This is because division requires an extra condition.

**Definition 6.4.2.** If \( f \) and \( g \) are functions, we define the **quotient** of \( f \) and \( g \) to be \( \frac{f}{g} \), where

\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},
\]

where \( x \) is in the domains of both \( f \) and \( g \) and \( g(x) \neq 0 \).

We require that \( g(x) \neq 0 \) to avoid a possible division by 0.

**Example 6.4.3.** Let \( f(x) = x + 1 \) and \( g(x) = x^2 - 4 \). Then the domains of \( f \) and \( g \) are both equal to the set of real numbers, but the domain of

\[
\left(\frac{f}{g}\right)(x) = \frac{x + 1}{x^2 - 4}
\]

is \( \{x | x \neq \pm 2\} \). The numbers we had to remove from consideration, \( \pm 2 \), were those numbers that make \( g \) equal to zero. We have

\[
\left(\frac{f}{g}\right)(3) = \frac{3 + 1}{3^2 - 4} = \frac{4}{5}.
\]

\(\square\)

**Example 6.4.4.** Let \( f(x) = \sqrt{x} \) and \( g(x) = 2x - 4 \). The domain of \( f \) is \( \{x | x \geq 0\} \) and the domain of \( g \) is the set of all real numbers. The domain of

\[
\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{2x - 4}
\]

is \( \{x | x \geq 0 \text{ and } x \neq 2\} \). We must again remove any values that make the denominator equal to zero.

\[
\left(\frac{f}{g}\right)(4) = \frac{\sqrt{4}}{2(4) - 4} = \frac{1}{2}
\]

\[
\left(\frac{f}{g}\right)(8) = \frac{\sqrt{8}}{2(8) - 4} = \frac{\sqrt{2}}{6}.
\]
There is another kind of operation we can perform on functions. It is called **composition**.

**Definition 6.4.3.** If $f$ and $g$ are functions, we define the **composition** of $f$ and $g$ to be the function $f \circ g$ defined by

$$f \circ g(x) = f(g(x)),$$

where $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$.

**Example 6.4.5.** Let $f(x) = 2x + 1$ and $g(x) = x^2$. Then

\[
(f \circ g)(x) = f(g(x)) \\
= f(x^2) \\
= 2x^2 + 1.
\]

Also,

\[
(g \circ f)(x) = g(f(x)) \\
= g(2x + 1) \\
= (2x + 1)^2 \\
= 4x^2 + 4x + 1.
\]

First, notice that $f \circ g$ and $g \circ f$ are different; composition of functions is not commutative. Also, we again caution you to take function evaluation **literally**. When we have $f(x) = 2x + 1$, it means that no matter what $x$ is, we double it and then add 1 to find the corresponding output from $f$. To evaluate $f(x^2)$, then, we double $x^2$ and then add 1: $f(x^2) = 2x^2 + 1$.

\[
(f \circ g)(-2) = f(g(-2)) \\
= f((-2)^2) \\
= f(4) \\
= 2(4) + 1 \\
= 9.
\]

Think of composition of functions as a two-step process. First, evaluate $g$ at $x$. Then take the output from that, and substitute it into $f$. This is why the domain looks like it does: in evaluating $f \circ g$ at $x$, the first step is to evaluate $g$ at $x$, so $x$ had better be in the domain of $g$. The second step is to evaluate $f$ at the output from $g$, so that output (which is $g(x)$) had better be in the domain of $f$.

**Example 6.4.6.** Let $f(x) = \frac{1}{x}$. Compute $(f \circ f)(x)$ and $(f \circ f)(-2)$.
Solution:

\[
(f \circ f)(x) = f\left(\frac{1}{x}\right) = \frac{1}{x} = 1 \cdot \frac{x}{1} = x,
\]

provided that \(x \neq 0\). Thus \((f \circ f)(-2) = -2\) since \(-2 \neq 0\).

\[
\square
\]

**Example 6.4.7.** Let \(f(x) = \sqrt{x}\) and \(g(x) = 1 - x\). Then

\[
(f \circ g)(x) = f(g(x)) = f(1 - x) = \sqrt{1 - x}.
\]

We know that 2 is in the domain of \(g\); in fact, \(g(-2) = -1\). For that matter, 2 is in the domain of \(f\), as well: \(f(2) = \sqrt{2}\). However, 2 is *not* in the domain of \(f \circ g\). The reason is in the definition of the composition:

\[
(f \circ g)(2) = f(g(2)) = f(-1),
\]

which is not defined. The trouble is that the output from \(g\) is the input into \(f\), so the output from \(g\) must be in the domain of \(f\); in this case, it isn’t. In fact, then domain of \(f \circ g\) is \(\{x | x \leq 1\}\).

\[
(f \circ g)(0) = f(g(0)) = f(1 - 0) = \sqrt{1} = 1.
\]

\[
(f \circ g)(-3) = f(g(-3)) = f(1 - (-3)) = \sqrt{4} = 2.
\]

\[
\square
\]

Composition turns out to be a very important way of combining functions. In later classes, you may see just how important this idea is.
Section 6.4 Exercises

Each function \( h \) is described in terms of two other functions \( f \) and \( g \). Determine the domain of \( h \).

1. \( f(x) = x^2, g(x) = x + 1, h(x) = \frac{f}{g} \).
2. \( f(x) = \sqrt{x}, g(x) = \sqrt{2x}, h = f - g \).
3. \( f(x) = 7x + 5, g(x) = \sqrt{3x + 4}, h = f + g \).
4. \( f(x) = \sqrt{x + 1}, g(x) = 1 - x, h = f \circ g \).
5. \( f(x) = \frac{x + 1}{x - 1}, g(x) = x + 1, h = \frac{f}{g} \).
6. \( f(x) = \frac{1}{\sqrt{x}}, g(x) = x - 2, h = f \circ g \).

Perform the indicated operations and simplify. Find the domain for each function, if appropriate. (Remember that the domain of sum, difference, etc. must be found before simplifying.

7. \( f(x) = \frac{1}{x^2 + 1}, g(x) = \frac{1}{x - 1}, (f + g)(x), \left( \frac{f}{g} \right) (-5), (f - g)(4) \).
8. \( f(x) = x^2 - 3x + 5, g(x) = 3x + 4, h(x) = -2x^2 - 9, (f + g + h)(3), \left( \frac{f - h}{g} \right) (x) \).
9. \( f(x) = 2x - 3, g(x) = x^2 + 2, (f \cdot g)(-4), (f + g)(x) \).
10. \( f(x) = x^2, g(x) = x + 1, \left( \frac{f}{g} \right) (1), \left( \frac{f}{g} \right) (x) \).
11. \( f = \sqrt{x}, g = \sqrt{2x}, (f - g)(18), \left( \frac{g}{f} \right) (x) \).
12. \( f(x) = 7x + 5, g(x) = \sqrt{3x + 4}, (f + g)(-1), \left( \frac{g}{f} \right) (4), (f \circ g)(x) \).
13. \( f(x) = \sqrt{x + 1}, g(x) = 1 - x, (f \circ g)(-7) \).
14. \( f(x) = \frac{x + 1}{x - 1}, g(x) = x + 1, \left( \frac{f}{g} \right) (x) \).
15. \( f(x) = \frac{1}{\sqrt{x}}, g(x) = x - 2, (f \circ g)(15) \).
16. Let \( f(x) = x^2 \) and \( g(x) = 2x - 1 \).
   (a) Find \( (f \circ g)(x) \) (including its domain).
   (b) Find \( (g \circ f)(x) \) (including its domain).
(c) What is \((f \circ g)(-2)\)?

17. Let \( f(x) = \frac{1}{x - 1} \) and \( g(x) = \sqrt{x} \).

   (a) Find \((f \circ g)(x)\) (including its domain).
   (b) Find \((g \circ f)(x)\) (including its domain).
   (c) What is \((f \circ g)(4)\)? \((g \circ f)(3)\)?

18. Profit is defined as revenue (income) minus cost (expense). If a certain company’s revenue function is described by \( R(x) = -x^2 + 18x + 6 \) and its cost function is \( C(x) = 4x - 22 \), find its profit function \( P(x) \). (Here \( x \) is the number of units produced.)
Chapter 7

Solutions to Selected Exercises

Section 1.1

1. False, but 0 is a whole number.  
2. False. 
3. True 
4. True 
5. True 
6. True 
7. True 
8. True 
9. Equality is symmetric. 
10. Equality is reflexive. 
11. Equality is symmetric. 
12. Equality is transitive.

Section 1.2

NOTE: Any choice of variables is acceptable.

1. Let \( n \) be the number of petitions James files. His court fees are then \( 10n \) dollars.
2. Let \( n \) be the number of nights Kim stays in the hotel. Then her hotel bill is \((9.90+85)n\).
3. Let \( n \) be the number of nights Kim stays in the hotel. Then her hotel bill is \((9.90+85)n\).
4. Let \( A \) represent the area (in square feet) to be painted. To paint this much area, we need \( \frac{4}{400} \) gallons. (Multiply this number of gallons by 400 to see how much area it covers.)
5. Let \( n \) be the number of times Carlos uses an ATM. His fees are then \( 4.95 + 0.5n \), and we are told that this is 8.45 dollars. Solving gives \( n = 7 \) times.
6. Seven hours is \( 7 \cdot 60 = 420 \) minutes. If \( p \) is the number of pens Thadd made, then \( 15 + 45p = 420 \). Solving gives \( p = 9 \) pens.

Section 1.3

1. \( 8 \cdot 5 \)
2. \( 12 \cdot (3 + 5) \) ("The sum of three and twelve" is treated as a single number that must be multiplied by 12.)
5. $4(8 - 5)$

7. The product of 4 and 5.


11. The sum of 5 and the product of 9 and the sum of 8 and 2.

13. $5 \cdot (4 + 2) = 5 \cdot 6 = 30.$

15. $6 - [5 \cdot (7 + 1)] = 6 - [5 \cdot 8] = 6 - 40 = -34.$

17. $4.6 + (12.2 + 6.8) = 4.6 + 19 = 23.6.$

19. $41 - (18 + 6) = 41 - 24 = 17.$

21. $7 - 12 = 7 + (-12).$

23. $x - 5 = x + (-5).$

25. $117 - x = 117 + (-x).$

Section 1.4

1. 29

3. 4

5. 10

7. 10

9. 81

11. 4

13. $(4 \cdot 3) + 1$

15. $(5 - 3) - 4$

17. $(\{4 + [12 \div (2 \cdot 3)]\} - 3) - 4.$

19. $[(4 \cdot 6) \div 2] \cdot 11.$

Section 1.5

1. Additive identity.

3. Distributive law.

5. Additive inverse.

7. Multiplicative identity


11. Additive inverse.

13. The distributive property: $3(1.52) + 7(1.52) = (3 + 7)(1.52) = 10(1.52) = 15.20.$

15. The distributive property: $7x + 5x = (7 + 5)x = 12x.$

Section 2.1

In each problem, a strip is one unit.

1. [Diagram of a strip]

3. [Diagram of a strip]

5. [Diagram of a strip]
7. $0.25 = \frac{1}{4}$

9. $1.5 = \frac{3}{2}$

11. $\frac{3}{8}$

13. $\frac{5}{3}$

15. $\frac{13}{35}$

17. $45\%$

19. $32\%$

21. $22.7\%$

23. $\frac{14}{100}$

25. $\frac{138}{100}$

Section 2.2

1. (a) $\frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3}$. (b) $8(3) = 24 = 12(2)$.

3. (a) $\frac{9}{12} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{3}{4}$. (b) $9(4) = 36 = 12(3)$.

5. (a) $\frac{-4}{8} = \frac{-1 \cdot 4}{2 \cdot 4} = \frac{-1}{2}$. (b) $-4(2) = -8 = 8(-1)$.

7. (a) $\frac{28}{40} = \frac{14 \cdot 2}{20 \cdot 2} = \frac{14}{20}$. (b) $28(20) = 560 = 40(14)$.

9. (a) $\frac{-5}{-8} = \frac{5(-1)}{8(-1)} = \frac{5}{8}$. (b) $(-5)(8) = -40 = (-8)(5)$.

11. (a) $\frac{21}{15} = \frac{(-7)(-3)}{(-5)(-3)} = \frac{-7}{-5}$. (b) $21(-5) = -105 = 15(-7)$.

13. $12(10) = 120 = 15(8)$.

15. $12(12) = 144 = 16(9)$.

17. $30(14) = 420 = 35(12)$.

19. $(-4)(-15) = 60 = 15(4)$.

21. $(x + 1)(3) = 3(x + 1)(1)$.

23. Recall that $5 = \frac{5}{1}$. $5x(1) = x(5)$. 

25. 

27. 

29. $\frac{1}{4}$

31. $\frac{6}{10}$

33. $\frac{7}{10}$
31. Since $a = a \cdot 1$, $\frac{a}{ab} = \frac{a}{a} \cdot \frac{1}{b} = \frac{1}{b}$ by the Fraction Simplification Theorem.

Section 2.3

1. $\frac{2}{3}$
2. $\frac{2}{5}$
3. $\frac{4}{5}$
4. $\frac{9}{7}$
5. $\frac{9}{7}$
6. $\frac{9}{7}$
7. $\frac{9}{7}$
8. $\frac{9}{7}$
9. $\frac{3}{4}$
10. $\frac{12}{5}$
11. $\frac{12}{5}$
12. $\frac{12}{5}$
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98. $\frac{12}{5}$
99. $\frac{12}{5}$
100. $\frac{12}{5}$

Section 2.4

1. $\frac{2}{3}
2. \frac{2}{5}
3. \frac{4}{5}
4. \frac{9}{7}
5. \frac{9}{7}
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9. \frac{3}{4}
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99. \frac{12}{5}
100. \frac{12}{5}

Section 2.5

25. \(\frac{24}{x}, x \neq 0\)
27. \(\frac{x}{3}\)
29. \(\frac{1}{x + 2}, x \neq -1, -2\)
31. \(\frac{(4 + t)(t + 2) + (4 - t)}{(4 - t)(t + 2)}, t \neq -2, 1\)
33. \(\frac{(2x - 1)(x + 5) - 3x}{3(x + 5)}, x \neq -5\)
35. \(\frac{47 + z}{3z}, z \neq 0\)
37. 5 inches.
39. \(4\frac{3}{16}\)
Section 2.6

1. $x = \frac{5}{2}$

3. $x = \frac{24}{5}$

5. $x = 32$

7. $z = \frac{45}{4}$

9. $t = \frac{8}{5}$

11. $x = \frac{165}{14}$

13. 12.6 gallons.

15. 3.1831 meters.

17. 735 Newtons.

19. 19.2 cm.

21. 5 inches.

23. 100 feet.

25. $x = 3\frac{3}{4}$

27. $x = 15$.

Section 2.7
1. \( \frac{(x - 2)}{(x + 2)}, x \neq 0 \)
3. \( \frac{1}{3}, x \neq \frac{5}{3} \)
5. \( 4x - 3, x \neq \frac{1}{2} \)
7. \( \frac{-7}{4 - 3x}, x \neq 0 \)
9. No simplification is possible.
11. \( \frac{4}{7x - 1}, x \neq 0 \)
13. \( \frac{5}{2x} \)
15. \( \frac{8t + 5}{5} \)
17. \( \frac{x}{x + 1}, x \neq 2 \)
19. \( \frac{x - 3}{3x}, x \neq -\frac{2}{5} \)
21. \( \frac{2}{x + 1}, x \neq \frac{3}{2} \)
23. \( \frac{(5x + 24)(21x - 4)}{(x - 3)(x + 2)}, x \neq 0 \)
25. \( \frac{4x - 14}{3x}, x \neq 0 \)
27. \( \frac{70x + 90 - 2x}{x(12x + 15)} \)
29. \( \frac{-45t + 25}{24} \)
31. \( \frac{68x - 23}{(x + 3)(5x - 1)} \)
33. \( \frac{u + 1}{16(t + 1)}, t, u \neq 0 \)
35. \( -118x - 99 \)
37. \( \frac{40x - 81}{12} \)
39. \( \frac{(4x + 5)(7x - 1)}{3x + 5}, x \neq \frac{1}{7} \)
41. \( \frac{x - 5}{3(x + 5)}, x \neq 0 \)
43. \( \frac{17(x - 14)}{4(2x + 1)}, x \neq 0, 3 \)
45. \( \frac{(x + 4)(3x - 2)}{3}, x \neq \frac{1}{3} \)

Section 3.1

1. \( 2^4 \)
3. \( (2 + 3)^3 \)
5. \( 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32 \)
7. \( 2.5 \cdot 2.5 \cdot 2.5 = 15.625 \)
9. \( -(4 \cdot 4 \cdot 4) = -64 \)
11. \( 4 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 2916 \)
13. \( -5(-5)(-5)(-5) = 625 \)
15. \( (2a)(2a)(2a) = 8a^3 \)
17. \( (3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3) = 3^6 = 729 \)
19. \( (2^5)(2^7) = (2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^12 = 4096 \)
21. \( (3^2)^3 = (3^2)(3^2)(3^2) = (3 \cdot 3)(3 \cdot 3)(3 \cdot 3) = 3^6 = 729 \)
23. \( \left( \frac{x}{4} \right) \left( \frac{x}{4} \right) \left( \frac{x}{4} \right) = \frac{x^3}{4^3} = \frac{x^3}{64} \)
25. \( (3x)(3x) = 9x^2 \)
27. \( \left( \frac{x}{-2} \right) \left( \frac{x}{-2} \right) \left( \frac{x}{-2} \right) \left( \frac{x}{-2} \right) = \frac{x^4}{(-2)^4} = \frac{x^4}{16} \)
29. No.
31. Yes.
33. No.
D stands for Degree, LT stands for Leading Term, LC stands for Leading Coefficient, and CT stands for Constant Term.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>D</th>
<th>LT</th>
<th>LC</th>
<th>CT</th>
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<td>45.</td>
<td>1</td>
<td>$z$</td>
<td>1</td>
<td>5</td>
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</tbody>
</table>

53. (a), (c), (e), and (s) are like terms. (b) and (d) are like terms. (f), (i), and (o) are like terms. (g) and (m) are like terms. (h), (p), and (r) are like terms. (j) and (t) are like terms. (k) and (q) are like terms. (l) and (n) are like terms.

57. $2^3 \cdot 3$
59. $2^2 \cdot 5^2$
63. $2^4 \cdot 3 \cdot 7^2$
65. $5^3 \cdot 7^2$

**Section 3.2**

1. $4^{15}$
15. 1
29. $5x^{15}$
3. $\frac{3^9}{8^9}$
17. $\frac{(x^2 + 4)^3}{3x^3(x + 1)^3(x + 2)}$
31. $\frac{(t + 1)^3(t^2 + 1)^4}{4}$
5. $5^3$
19. $x^{48}y^{72}$
33. $\frac{3^5y^4}{z^2(x + 2)^2}$
7. $(3^{28})$
21. $-\frac{5^5x^5}{8^5}$
35. $\frac{11x^2 + 2x}{(2x + 1)(x - 3)}$
9. $16x^4$
23. $a^5b^5c^5d^5e^5$

39. $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3)$, while $9^6 = 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$. The first has 6 factors of 3, the second has 6 factors of 9.

**Section 3.3**

1. $\frac{1}{3^3}$
3. $3^4$
7. $\frac{1}{14^{11}}$
5. $4x^2$
9. \( \frac{5}{4^3} \)  
11. \( \frac{1}{t^4} \)  
13. \( \frac{1}{2^7} = 2^{-7} \)  

15. \( 4^{-15} = \frac{1}{4^{15}} \)  
17. \( x^{-2} = \frac{1}{x^2} \)  
19. \( x^{20} y^5 \)  

21. \( \frac{x^{-8}}{y^{-6}} = \frac{y^6}{x^8} \)  
23. \( \frac{1}{x^6 y^5} = x^{-6} y^{-3} \)  

Section 3.4

1. 2  
3. 3  
5. 32  
7. 4  

9. \( \frac{3}{2} \)  
11. \( \frac{8}{27} \)  
13. \( x^2 \)  
15. \( 343x^9 \)  

17. \( y^\frac{3}{2} \)  
19. \( x^{\frac{1}{20}} \)  
21. \( x^{\frac{3}{2}} y^{\frac{1}{2}} \)  
23. \( x^{\frac{1}{20}} y^{\frac{1}{160}} \)  

Section 3.5

13. \( \sqrt[3]{3^4} \)  
15. \( \sqrt[7]{x^4} \)  
17. \( \sqrt[2]{x^2} \)  
19. \( \sqrt[3]{x^2} \)  
21. \( \sqrt[7]{x^7} \)  

23. \( \sqrt[3]{x^3} \)  
25. \( x^{\frac{5}{2}} \)  
27. \( x^{\frac{5}{2}} \)  
29. \( \frac{x^{\frac{7}{2}}}{y^{\frac{5}{2}}} \)  

31. \( x^{\frac{4}{2}} \)  
33. \( \frac{4^4 y^{\frac{2}{3}}}{x^{\frac{7}{3}}} \)  
35. \( 2 x^{\frac{11}{2}} \)  

37. Radicand: \( x^6 \). Index: 5. \( x^{\frac{5}{2}} \sqrt{x} \).  
43. Radicand: \( 15 x^3 \). Index: 2. \( x\sqrt{15x} \).  
39. Radicand: \( 25 x^4 \). Index: 2. \( 5 x^{2} \sqrt{x} \).  
45. Radicand: \( x^{11} \). Index: 4. \( x^2 \sqrt[3]{x^3} \).  
41. Radicand: 5. Index: 2. \( \sqrt{5} \).  
47. Radicand: \( y^0 \). Index: 2. \( y^4 \sqrt{y} \).  

49. \( 4 x^4 y^2 \sqrt{3y} \)  
51. \( 3 t^3 \sqrt{6} \)  

53. \( x^7 \)  
55. \( \frac{\sqrt{x^2 + 1}}{4x^2} \)  

57. \( \frac{3 \sqrt[3]{3}}{x \sqrt[3]{x^2}} \)  
59. \( 10 \sqrt[3]{3} \)  

Section 3.6

1. \( 3.412 \times 10^2 \)  
3. \( 5.8143 \times 10^4 \)  
5. \( 8.3 \times 10^{-6} \)  

7. \( 1.683 \times 10^0 \)  
9. \( 4.69 \times 10^{17} \)  
11. \( 7.9 \times 10^6 \)  

13. \( 1.71 \times 10^{10} \)  
15. 4600  
17. 0.06315
19. 6.4
21. 0.0823
23. 0.99995
25. 551000000000

Section 4.1

1. $4 \cdot 7 + 5 \cdot 7$
23. $4x^7 - 8x^3 + 3x^7 + 2$
3. $-4x + (-4)(3)$
25. $2x(x + 2)$
5. $4x^2 + 2x$
27. $4x^2(x + 2)$
7. $(x^7 + 2x^5 - 3x^3)$
9. $-2t^3 - 8t^4$
11. $\frac{1}{10}x^4 - \frac{8}{15}x^3 + \frac{7}{20}x^2 - \frac{2}{5}x$
31. $(x - 3)(x + 4)$
13. $6x^2 + 8x$
33. $(x + 1)x^\frac{3}{2}$
15. $-3x^4$
35. $(x^2 - 1)y^2$
17. $-54x$
37. $5x^2 + 8x - 1$
19. $4x^7 + 7x^3$
39. $3x^2 + 4x + 4$
21. $\frac{1}{24}x^2 + \frac{17}{4}x$
41. $3x^3 + x^2 + 3x$

Section 4.2

1. $-x - 3$
17. $-5x^6 - x^5 - x^3$
33. $-\sqrt{x^2 + 4}(1 + 6x)$
3. $-3x + 5$
19. $x(x - 1)$
35. $(3x^2 - 5x + 4)\sqrt{x}$
5. $-x^5 + 4x^3$
21. $-x^3(x + 4)$
1. $(9x^3 + 21x^2 + 3x + 15)$
7. $7x^2 - 4x$
23. $3x(3x^2 + 4x + 7)$
9. $t - 3$
25. $-(w + 1)$
3. $22x$
11. $\frac{1}{4}x^5 + \frac{1}{2}x^4 + \frac{1}{4}x^3$
27. $-2(x + 1)$
5. $-x^6 - 4x^3$
13. $3x^\frac{6}{5} - 7x^\frac{3}{5}$
29. $-x^\frac{1}{3}(x^\frac{1}{3} + 1)$
7. $4x^3 + 9x^2 + 2x - 2$
15. $4x^4 - 28x^3 + 12x^2 + 8x$
31. $3\sqrt{x + 1}$
9. $5x^3$

Section 4.3

Draw a diagram like that of Figure 4.2 to illustrate each computation.
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1. \( x^2 + 5x - 36 \)
2. \( x^2 + 10x + 24 \)
3. \( x^2 - 8x + 12 \)
4. \( t^2 - 2t - 8 \)
5. \( x^2 + 1 - \frac{3}{8} \)
6. \( 6x^2 - 10x - 4 \)
7. \( t^2 + 4t + 2 \)
8. \( 9x^2 - 6x + 1 \)
9. \( x + 2\sqrt{x} + 1 \)
10. \( x^2 - 36 \)
11. \( x^2 - 2 \)
12. \( 2x^2 - 20x - 33 \)
13. \( 3x^2 + 5x - 2 \)
14. \( x^2 - 1 \)
15. \( x^2 - 64 \)
16. \( 9x^2 - 4 \)
17. \( 121x^2 - 16 \)
18. \( x^2 - a^2 \)
19. \( x^2 + 10x + 25 \)
20. \( x^2 + 12x + 36 \)
21. \( x^2 + \frac{1}{2}x + \frac{1}{16} \)
22. \( 25x^2 - 70x + 49 \)
23. \( x^3 - x^2 - 5x + 2 \)
24. \( x^5 - 2x^2 - x - 2 \)
25. \( 6x^4 + 22x^3 + 43x^2 + 43x + 21 \)
26. \( -21x^4 - 29x^3 - 58x^2 + 37x + 33 \)

**Section 4.4**

1. \( x^2 + 2x + 1 \)
2. \( t^2 + 4t + 2 \)
3. \( 9x^2 - 4 \)
4. \( 49x^2 - 25 \)
5. \( x^2 + \frac{8}{3}x + 16 \)
6. \( x + 4\sqrt{x} + 4 \)
7. \( x^4 - 1 \)
8. \( x^2 - 36 \)
9. \( x^2 - 2 \)
10. \( (x + 2)^2 \)
11. \( (x + 3)^2 \)
12. \( (2x + 5)^2 \)
13. \( (x - 1)(x + 1) \)
14. \( (x - 4)(x + 4) \)
15. \( (x - 9)(2x + 9) \)

**Section 4.5**

1. Neither. \((x + 4)(x + 1)\)
2. PST. \((x - 4)^2\)
3. PST. \((x + 8)^2\)
9. Neither. \((x + 4)(x - 1)\)
11. Neither. \((x + 5)(x + 2)\)
13. DOS. \((x - 11)(x + 11)\)
15. Neither. \((x + 16)(x + 4)\)
17. Neither. \((x - 8)(x + 5)\)
19. DOS. \((3x - 7)(3x + 7)\)
21. PST. \((4x + 5)^2\)
23. Neither. \((2x - 1)(5x + 1)\)
25. Neither. \((7x - 2)(3x + 5)\)
27. Neither. \((-2x + 1)(4x + 7)\)
29. Neither. \((5x + 2)(2x + 1)\)
31. Neither. \((2x + 1)(4x - 3)\)
33. Neither. \((x + 1)(4x + 5)\)
35. Neither. \((6x + 3)(3x + 2)\)

Section 5.1

1. \(x = -3\)
3. \(x = 11\)
5. \(x = 4\)
7. \(x = -2\)
9. \(x = 5\sqrt{5}\)
11. \(x + 15 = -\frac{49}{3}\)
13. \(x = -3\)
15. \(x = \frac{12}{5}\)
17. \(x = -1\)
19. \(x = -\frac{3}{2}\)
21. \(x = \frac{88}{3}\)
23. \(x = -\frac{8}{5}\)
25. \(x = 8\)
27. \(x = \frac{11}{5}\)
29. \(x = -\frac{1}{14}\)
31. 19 books.
33. About $51818.18.
35. \(x = 2\) or \(x = 3\).
37. \(x = -9\) or \(x = 7\).
39. \(x = -\frac{1}{3}\) or \(x = \frac{1}{2}\).

Section 5.2

1. 
3. 
7. \(\{x | x > -3\}\)
9. \(\{x | x < -4\}\)
11. \(\{x | x \geq 1\}\)
13. \(\{x | x > -1\}\)
15. \(\{x | x \leq -5\}\)
Section 5.3

1. \((4.75, 3.5)\)  
5. \((-1, 2)\)  
9. \((-1.5, -1)\)  
13. \((2, -2)\)

3. \((2, 4)\)  
7. \((-3, 1)\)  
11. \((-0.75, -2.5)\)  
15. \((4, -1)\)

21. \(\{x | x > \frac{7}{3}\}\)  
25. \(\{x | x < \frac{8}{7}\}\)

19. \(\{x | x \geq \frac{11}{5}\}\)  
23. \(\{x | x \leq -\frac{25}{4}\}\)

21. \(\{x | x > -\frac{3}{2}\}\)  
27. \(\{x | x \leq -7\}\)

23. \(\{x | x \leq -\frac{25}{4}\}\)  
31. After 87.5 miles.

33. After 350 minutes (5 hours and 50 minutes).
Section 5.4

1. Negative.
3. Positive.
5. Negative.
7. Undefined.
11. \( \frac{5}{9} \)
13. \( -\frac{6}{5} \)
15. 0.
12. \( -\frac{2}{5}, P = (1, 4) \)
14. \( m = 1; P = (0, 0) \)
16. \( m = -3 \)
18. \( m = 0 \)
20. \( m \) is undefined.
22. \( \frac{7}{2} \)
24. \( m = 0 \)
26. \( m \) is undefined.
28. \( a = 5, b = -\frac{1}{5} \)
30. \( a = -\frac{5}{2}, b = \frac{2}{5} \)
32. \( a = -8, b = \frac{1}{8} \)
34. \( a = -\frac{7}{4}, b = \frac{4}{7} \)

Section 5.5

1. \( m = 4, b = 2 \)
3. \( m = 1, b = -1 \)
5. \( m = 0, b = 9 \)
7. \( y = -\frac{1}{2}x - 2 \)
9. \( y = x + 3 \)
11. \( y = -4.5 \)
13. \( y = 4x - 2 \)
15. \( y + 4 = \frac{4}{5}(x - 3) \)
17. \( y - 4 = \frac{1}{3}(x + 1) \)
19. \( y = \frac{10}{21}x \)
21. \( x = 6 \)
23. \( y + 2 = -\frac{5}{7}(x + 9) \)
25. \( y - 5 = \frac{5}{4}(x + 1) \)
27. \( m = -\frac{1}{3} \)
29. \( m = 5 \)
31. \( m = -\frac{6}{5} \)
33. \( y = 18 \)
35. \( y = \frac{6}{7} \)
37. \( y = -\frac{117}{16} \)

Section 5.6
Find the x- and y-intercepts of each line.

1. x-intercept: \(-\frac{15}{2}\).
y-intercept: 5.
3. x-intercept: \(\frac{5}{2}\).
y-intercept: 2.
5. x-intercept: 3.
y-intercept: \(\frac{1}{2}\).
7. \( 2 = \frac{2}{3}x + 5 \)
9. \( 4x + 5y - 10 = 0 \)
11. \( 2x + 12y - 6 = 0 \)
13. \( y = 2x + 8 \)
15. \( y + 1 = \frac{5}{2}(x - 4) \)
17. \( y + 6 = \frac{4}{5}x \)
19. \( x - 3 = 0 \)
21. \( y = -\frac{5}{2} \)
23. \( -2x + 4y - 6 = 0 \)
Section 6.1

1. Function.

3. Wrong range: not all outputs will be integers.

5. Wrong range: not all outputs will be natural numbers.

7. Wrong domain: not all real numbers are allowable inputs. For example, $\sqrt{-1}$ is not a real number. (OR: Wrong range. Ask your teacher for clarification.)

9. Wrong domain: 0 is not an allowable domain element.

11. Wrong range: Not all outputs are rational. OR: Wrong domain: restricting the domain to rational numbers would work.

13. 5, 2, 2, 5, 26. (in order)

15. $\frac{2}{3}, \frac{5}{3}, 4, 33$.

17. $\frac{1}{5}, 0, \frac{3}{13}, \frac{7}{37}$.

19. $\frac{1}{4}, 1, 3, x$ (if $x \neq 0$), $\frac{1}{12}$.

21. 3, 3, 1, $(t - 3)^4 + (t - 3)^2 + 1$.

23. 2, 2, 5, 5, 10, 10.

25. The set of all real numbers.

27. The set of all real numbers.

29. \(\{x|x \neq -1/3\}\).

31. \(\{x|x \geq -\frac{5}{3}\}\).

33. \(\{x|x \neq \frac{2}{3}\}\).

35. \(\{x|x \neq 2, 4\}\).

Section 6.2

1. \(\{x|x \text{ is a real number}\}\). 1. Polynomial. 7. $-19$

3. \(\{x|x \leq 2\}\) 3. Radical. 9. $2\sqrt{5}$

5. \(\{x|x \neq -2, 5\}\) 5. Radical. ($x^{\frac{2}{3}} = \sqrt[3]{x^2}$). 11. $-\frac{1}{12}$
13. $-7$

15. $\sqrt{3}$

17. $1024$

19. $25, 25, 9x^2, x^2 - 2x + 1.$

21. $0, \frac{\sqrt{3}}{2}$

Section 6.3

1. $f(-5) = 5, f(-2) = 2, f(1) = -1, f(2) = 3, f(4) = 5$

3. $f(-3) = -5, f(-2) = -3, f(-1) = -1, f(2) = -4$

5. $f(-2) = 7, f(1) = 1, f(4) = 5, f(7) = 11$

7. $f(-2) = 0, f(0) = -2, f(2) = 0, f(5) = 3$

$f(x) = |x - 1|$

$f(x) = |2x|$
5. \[ f(x) = |x| + 1 \]

7. Let \( C \) represent the cost of the rental, and let \( x \) represent the number of miles driven. Then
\[
C(x) = \begin{cases} 
24.95 & \text{if } x \leq 200 \\
24.95 + 0.35(x - 200) & \text{if } x > 200.
\end{cases}
\]
\[ C(50) = 24.95, \quad C(150) = 24.95, \quad C(250) = 24.95 + 0.35(250 - 200) = 42.45, \quad \text{and} \quad C(350) = 24.95 + 0.35(350 - 200) = 77.45. \]

9. A mail-order book company advertises shipping rates of $5 for up to 6 books, and then $0.75 for each additional book.

(a) The domain is the set of natural numbers.

(b) \[ f(x) = \begin{cases} 
5 & \text{if } x \leq 6 \\
5 + 0.75(x - 6) & \text{if } x > 6
\end{cases} \]

(c) \[ f(3) = 5 \text{ and } f(12) = 5 + 0.75(12 - 6) = 9.50. \]

**Section 6.4**

Each function \( h \) is described in terms of two other functions \( f \) and \( g \). Determine the domain of \( h \).

1. \( \{x|x \neq -1\} \)

3. \( \{x|x \geq -\frac{4}{3}\} \)

5. \( \{x|x \neq \pm 1\} \)

7. \((f + g)(x) = \frac{1}{x^2 + 1} + \frac{1}{x - 1}\) has domain \( \{x|x \neq 1\} \). \( \left(\frac{f}{g}\right)(-5) = -\frac{5}{39} \). \( (f - g)(4) = \frac{20}{51} \).

9. \((f \cdot g)(-4) = -198. \quad (f + g)(x) = x^2 + 2x - 1\) has domain the set of all real numbers.

11. \((f - g)(18) = 3\sqrt{2} - 6. \quad \left(\frac{g}{f}\right)(x) = \sqrt{2}\) with domain \( \{x|x > 0\} \).

13. \((f \circ g)(-7) = 3. \)

15. \((f \circ g)(15) = \frac{1}{\sqrt{13}}. \)
17. Let \( f(x) = \frac{1}{x - 1} \) and \( g(x) = \sqrt{x} \).

(a) \( (f \circ g)(x) = \frac{1}{\sqrt{x} - 1} \) has domain \( \{x | x \geq 0 \text{ and } x \neq 1\} \).

(b) \( (g \circ f)(x) = \sqrt{\frac{1}{x - 1}} \) has domain \( \{x | x > 1\} \).

(c) \( (f \circ g)(4) = 1 \). \( (g \circ f)(3) = \frac{1}{\sqrt{2}} \).